Importance Sampling for Integrated Market and Credit Portfolio Models

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Abstract:

The predominant approach in risk management is to determine the economic capital for each risk type separately. Thus, the problem arises how to combine these different amounts of capital to a single number. Beside just adding the economic capital values or assuming multivariate normality of the different risk types, the usage of Copulas has been proposed recently for linking the marginal distributions of losses. In this paper, a different approach is pursued by modeling market and credit risk simultaneously, whereby stochastic dependencies between these two risk types can be taken into account directly. However, integrating market risk factors into standard credit portfolio models increases the computational burden of calculating risk measures. That is why the application of various importance sampling techniques to an integrated market and credit portfolio model is presented. The computational difficulties which result from the additional integration of market risk are discussed. The effectiveness of these approaches is tested by numerical experiments for linear and non-linear portfolios.

Keywords: risk management, credit risk, importance sampling, interest rate risk, Value-at-Risk

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1 Introduction

Banks are exposed to many different risk types due to their business activities. Among these risk types are credit risk, market risk, operational risk, and business risk. The task of the risk management division is to measure all these risks and to determine the necessary amount of economic capital which is needed as a buffer to absorb unexpected losses associated with each of these risks. Most frequently, economic capital is understood as a Value-at-Risk (VaR) number. Thus, it is the amount of capital needed to absorb unexpected losses within a given time period up to a specified probability.

Predominantly, the necessary amount of economic capital is determined for each risk type separately. That is why the problem arises how to combine these various amounts of capital to a single number. Within the so-called building-block approach stipulated by the regulatory authorities, the amount of regulatory capital, which the banks have to hold for the different risk types, are just added. This is a quite conservative approach because it ignores diversification effects between the risk types. As a consequence, in general, the true amount of economic or regulatory capital, which is needed, is overestimated.

However, the alternative, namely to consider diversification effects to some extent, requires to model the multivariate dependence between the various risk types. In practice, some kind of heuristics, based on strong assumptions, are often used to merge the economic capital figures for the various risk types into one overall economic capital figure.[2] A theoretical much more sound approach is to link the separately determined marginal distributions of losses resulting from different risk types by adequate Copula functions (see, e.g., Ward and Lee (2002), Dimakos and Aas (2004), and Rosenberg and Schuermann (2006)). However, the difficulty is to choose the correct Copula function, especially given the limited access to adequate data.

Another approach is to build up models for various risk types by integrating a specific risk type into existing models for the measurement of another risk type. This approach is pursued in this paper, which deals, more specifically, with the integration of market risk into credit

[2] For an overview on risk aggregation methods used in practice, see Joint Forum (2003), Bank of Japan (2005), and Rosenberg and Schuermann (2006).

portfolio models. Integrated market and credit portfolio models allow to determine simultaneously, in a common framework, the necessary amount of economic capital needed for the market risk and for the credit risk of banking book instruments, whereby possible stochastic dependencies between these two risk components can be taken into account. This latter approach is called a *bottom-up* approach, whereas the copula-based approach represents *top-down* techniques

For measuring the credit risk inherent in a banking book, a range of models has been developed. Well-known examples are CreditMetrics by J.P. Morgan Chase, CreditPortfolioView by McKinsey, Portfolio Manager by KMV, or CreditRisk$^+$ by Credit Suisse First Boston. A typical shortcoming of most credit portfolio models is that relevant market risk factors, such as risk-free interest rates or credit spreads, are not modeled as stochastic variables and hence are ignored during the revaluation of the credit sensitive instruments at the risk horizon. An exception is the approach Algo Credit developed by the risk management firm Algorithmics (see Iscoe et al. (1999)). Even the Basel II proposals do not intend to regulate the interest rate risk of the banking book in a quantitative way, but only qualitatively under pillar II (see Basel Committee on Banking Supervision (2005)). In a typical credit portfolio model, fixed-income instruments, such as bonds or loans, are revalued at the risk horizon using the current forward rates and (rating class specific) forward credit spreads for discounting future cash flows. Even for derivatives with counterparty risk, only single values, so-called loan equivalents, are employed per possible rating grade of the counterparty at the risk horizon. Thus, the stochastic nature of the instrument's value in the future which results from changes in factors other than credit quality is ignored, which may underestimate the riskiness of the credit portfolio (see, e.g., Barnhill and Maxwell (2002), Kiesel et al. (2003), Grundke (2005)). An additional consequence is that correlations between changes of the debtors' credit quality and changes of market risk factors and hence the exposure at default cannot be integrated into the credit portfolio model. This is especially a problem for market-driven instruments, such as interest rate derivatives, because the exposure at default mainly depends on the stochastic evolution of the underlying market risk factors. Finally, ignoring relevant market risk factors in credit portfolio models, correlations between the exposures at default of different instruments, which depend on the same or correlated market risk factors, cannot be modeled, either.

However, adding market risk factors as additional ingredients of a credit portfolio model, the computational burden of calculating risk measures increases because the revaluation of the instruments at the risk horizon becomes more complex. Most standard credit portfolio models rely on Monte Carlo simulations for calculating the probability distribution of the future credit portfolio value.[3] This is already computer-time-consuming for standard credit portfolio models, especially for portfolios with many obligors and when percentiles corresponding to high confidence levels have to be estimated. Thus, the need of efficient methods for calculating credit risk measures becomes even more pressing for integrated market and credit portfolio models.

For standard credit portfolio models, various efficiency enhancing computational approaches have been developed meanwhile. Among these are, for example, approaches based on Monte Carlo simulations combined with variance reduction techniques, mainly importance sampling (IS) (for a literature review see the next section 2), Fourier-based approaches (see Duffie and Pan (2001), Merino and Nyfeler (2002), Reiß (2003)), computational approaches based on saddlepoint approximations (see, e.g., Arvanitis et al. (1998), Martin et al. (2001a, 2001b), Gordy (2002), Barco (2004)), or methods which rely on the assumption that the portfolio is sufficiently large or sufficiently granular so that by the virtue of the (strong) Law of Large Numbers (or the Central Limit Theorem) approximations of the credit portfolio loss variable are possible (see, e.g., Finger (1999), Vasicek (1991, 2002), Gordy (2003)). This paper fits best into the first category because it makes use of importance sampling as a special variance reduction technique when simulating the credit portfolio value at the risk horizon. Monte Carlo simulations combined with variance reduction techniques are reported to be very flexible in the computation of overall risk measures as well as individual risk contributions.

The paper is structured as follows: in section 2, an overview on related literature is given. In section 3, a general framework for an integrated market and credit portfolio model is presented. Besides, a concrete specification of this general model is described, which afterwards is used for the numerical experiments. In section 4, two and three step-IS techniques when applied to the general integrated market and credit portfolio model are discussed. The effectiveness of the presented IS techniques is tested by means of numerical experiments in section 5. Finally, in section 6, the main results are summarized and possible extensions of this study are outlined.

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3 A prominent exception is the model CreditRisk$^+$ where due to specific assumptions the loss distribution can be computed by recursion.

2 Review of the Literature

Most approaches based on Monte Carlo simulations combined with variance reduction techniques employ IS to accelerate the computation of credit risk measures. For a CreditMetrics-style, pure default mode credit portfolio model, Glasserman and Li (2005) employ IS for the probability distribution of the systematic risk factors as well as for the conditional default probabilities to estimate excess probabilities more efficiently than with naïve Monte Carlo simulations. In contrast to almost all other approaches, Glasserman and Li (2005) employ a two step-IS procedure. The same technique is applied to a CreditRisk$^+$-style credit portfolio model by Glasserman and Li (2003). Glasserman (2006) shows how this approach can be employed for estimating obligor-specific credit risk contributions. Dunkel and Weber (2005) employ the same approach as Glasserman and Li (2005) for estimating utility-based shortfall risk measures in the framework of the credit portfolio model CreditRisk$^+$ and the default mode version of CreditMetrics.

Merino and Nyfeler (2004) only use IS for the conditional default indicators, but leave the probability distribution of the systematic risk factors unchanged. The technique they employ is called 'exponential twisting' (see section 4.1 and Glasserman (2004, pp. 260)), which is also applied by Glasserman and Li (2005). They demonstrate the use of this technique for estimating individual risk contributions (based on the expected shortfall) within a classical 'conditional independence'-framework with default mode. In a numerical example, they choose a specification of the credit portfolio model in the spirit of CreditRisk$^+$.

Another branch of papers solely deals with IS for the systematic risk factors. These papers mainly differ in the way how an effective IS distribution for the systematic risk factors is determined. Within the framework of the default mode CreditMetrics model, Kalkbrener et al. (2004) try to find optimal means of the systematic credit risk factors under the IS distribution by approximating the original inhomogeneous, finite portfolio by a homogeneous, infinitely granular portfolio. Of course, this approximation procedure is not unique. Then, they calculate the mean of the systematic credit risk factor that minimizes the variance of the estimator of the desired risk measure in a one-factor model of the homogeneous, infinitely granular portfolio. Finally, they 'lift' this one-dimensional optimal mean to a M-dimensional mean vector. They use this technique for estimating individual risk contributions based on the expected shortfall. An approximation of the original portfolio by an infinitely granular, homogeneous portfolio is also employed by Tchistiakov et al. (2004) to reduce the variance of the risk measure estimator. However, they employ this approximation as a control variate.[4] In a default mode version of CreditMetrics, Egloff et al. (2005) compute the means of the systematic risk factors under the IS distribution by an adaptive stochastic approximation procedure. Instead of modifying the means of the systematic risk factors, Morokoff (2004) scales up the variances and covariances of the asset returns. For this, he orthogonalizes the original covariance matrix of the asset returns. Under the IS probability measure, the asset returns are sampled according to a normal distribution with mean zero and a modified covariance matrix, which results from scaling up the largest eigenvalue from the eigenvalue-eigenvector decomposition of the original covariance matrix. Morokoff (2004) works in a default mode version of CreditMetrics.

Bassamboo et al. (2006) also employ a 'conditional independence'-framework with default mode, but – in contrast to other papers – they allow for an extremal dependence between the obligors, which is for example induced by modeling the asset returns with a multivariate t-distribution. They propose two ways for deriving an IS distribution for that random variable which causes common shocks for all obligors and hence dependence amongst them. As Bassamboo et al. (2006) also suggest to use 'exponential twisting' for the conditional default probabilities, their approach is another two step-IS technique.

Joshi (2004) demonstrates for the pricing of collateralized debt obligations how to apply IS by shifting the mean of the systematic risk factor. Joshi and Kainth (2004) also use this technique for computing the sensitivity of the price of a n^{th} to default swap to changes in the underlying hazard rate of a particular obligor. These deltas are needed for hedging n^{th} to default swaps.

In this paper, it is shown in detail how the two step-IS technique presented by Glasserman and Li (GL) (2005) for a pure default mode model can be applied to a general integrated market and credit portfolio model. It is also pointed out which differences and difficulties result from the integration of market risk. As already mentioned, GL (2005) employ IS for the probability distribution of the systematic risk factors as well as for the conditional default probabilities. Due to this two step-IS technique, the approach suggested by GL (2005) is expected to be especially effective. That is why it is employed in this paper. Furthermore, it is discussed how an IS approach originally developed by Glasserman et al. (2000) for pure market risk portfolio

[4] Arvanitis et al. (1998) and Arvanitis and Gregory (2001, pp. 83) also work with a control variate, which is, however, based on saddlepoint approximations.

models can be combined with the two step-IS approach to build up a potentially even more effective three step-IS technique. Glasserman et al. (2000) use a delta-gamma approximation of the loss variable of a portfolio of default risk-free instruments for selecting an effective IS distribution for the normally distributed vector of market risk factors. As the topic of this paper are integrated *market* and credit portfolio models, the idea to combine methods originally developed for pure market risk portfolio models with those originally developed for pure (default mode) credit portfolio models might suggest itself. However, up to now, this has not been tried.

Summarizing, the main questions answered in this paper are:

1) Are IS techniques originally developed for pure default mode credit portfolio models also applicable to integrated market and credit portfolio models?
2) How effective are they for these extended models?
3) Is it possible to increase the effectiveness by combining IS techniques originally developed for pure default mode credit portfolio models with those originally developed for pure market risk portfolio models ?

3 The Integrated Market and Credit Portfolio Model

3.1 General Approach

It is assumed that the credit portfolio consists of N market and credit risk sensitive instruments issued by N different corporates. The risk horizon of the credit portfolio model is denoted by H. P denotes the real world probability measure. The number of possible credit qualities at the risk horizon is K: one denotes the best rating and K the worst rating, the default state. For default mode credit portfolio models, K is equal to 2, whereas for mark-to-market credit portfolio models, K is larger than 2.

The central part of most credit portfolio models is the definition of the obligors' conditional default and transition probabilities. Denoting by $\eta_H^n \in \{1,\ldots,K\}$ the credit quality of obligor n at the risk horizon H and by η_0^n the respective rating at $t=0$, the conditional default (transition) probabilities are formally defined as:

$$P\left(\eta_H^n = k \middle| \eta_0^n = i, Z_1 = z_1,\ldots,Z_C = z_C\right) := f_{n,i,k}(z_1,\ldots,z_C) \qquad (1)$$

with

$$f_{n,i,k}:\mathbb{R}^C \rightarrow [0,1] \; (k \in \{1,\ldots,K\},\; i \in \{1,\ldots,K-1\},\; n \in \{1,\ldots,N\}).$$

The set of variables $\mathrm{Z}=(Z_1,\ldots,Z_C)\sim G^C$ are systematic credit risk factors, which might be thought of as changes in equity indices or macro-economic variables until the risk horizon. They influence the credit quality changes of all obligors. This vector is assumed to evolve according to a multivariate distribution G^C. Given the realization $(Z_1=z_1,\ldots,Z_C=z_C)$ of the systematic credit risk factors and hence of the conditional default (transition) probabilities, credit quality changes of all obligors are assumed to be stochastically independent. Thus, this is the classical 'conditional independence'-framework for describing joint credit quality changes in a credit portfolio.

The price of the instrument i_n (e.g., a defaultable (zero) coupon bond or an option with counterparty risk) at the risk horizon H, whose issuer n has not already defaulted before H and exhibits the rating $\eta_H^n \in \{1,\ldots,K-1\}$, is denoted by

$$p_n(\eta_H^n;\mathrm{X};\mathrm{P_n}). \tag{2}$$

The stochastic vector $\mathrm{X}=(X_1,\ldots,X_M)\sim G^M$ represents the value of relevant market risk factors at the risk horizon. This vector is assumed to evolve according to a multivariate distribution G^M. $\mathrm{P_n}$ denotes a vector of additional parameters relevant for the pricing of the respective instrument i_n at the risk horizon. Note that the set of systematic credit risk factors $Z_1,\ldots,Z_C$ and the set of market risk factors $X_1,\ldots,X_M$ can overlap, e.g., if a risk-free interest rate is also a relevant credit risk driver. The joint distribution of the stochastic vector $(Z_1,\ldots,Z_C;X_1,\ldots,X_M)$ is denoted by G. Later for the numerical examples, it will be assumed that G is a multivariate normal distribution.

If the issuer n of the instrument i_n has already defaulted ($\eta_H^n=K$) until the risk horizon H, its value, in the case this value is positive, is set equal to a fraction δ_n of the value that the instrument would have at the risk horizon when its issuer would be free of default risk. If the market value of this instrument is negative, nothing is changed because the bank whose credit portfolio is considered is a debtor of the defaulted issuer. The recovery rate δ_n can vary with the seniority of a claim and the value of individual collaterals. Usually, it is assumed that the recovery rate is beta-distributed and independent from all other stochastic variables of the respective model, such as the systematic credit risk drivers or the market risk factors, but it could also be a function of these risk factors (see, e.g., Frye (2000), Pykhtin (2003)). Finally, the value $\Pi(H)$ of the entire portfolio at the risk horizon H, whose probability distribution is sought, is just the sum of the individual values:

$$\Pi(H) = \sum_{n=1}^{N} p_n(\eta_H^n; \mathrm{X}; \mathrm{P_n}) . \tag{3}$$

The credit portfolio loss variable is defined as the sum of the difference between the instrument's value at the risk horizon, when the initial rating of the obligor has not changed and the market risk variables equal their expected values, and the actual instrument's value at the risk horizon:

$$L(H) = \sum_{n=1}^{N} L_n(H) = \sum_{n=1}^{N} \left(p_n(\eta_0^n; E^P[\mathrm{X}]; \mathrm{P_n}) - p_n(\eta_H^n; \mathrm{X}; \mathrm{P_n}) \right).$$

Industry standards, such as the well-known credit portfolio models CreditMetrics, CreditPortfolioView, or CreditRisk$^+$, can be seen as special cases of the general modeling approach described above. All of these models have in common that stochastic fluctuations of market risk factors are not considered for the revaluation of the instruments at the risk horizon:

$$p_n(\eta_H^n; \mathrm{X}; \mathrm{P_n}) = p_n(\eta_H^n; \mathrm{P_n}) \quad \forall\, n \in \{1, \ldots, N\} . \tag{4}$$

3.2 A Special Case: CreditMetrics with Integrated Correlated Interest Rate Risk

As an example of the general integrated market and credit portfolio model described before, in this section, the usual CreditMetrics framework is extended by interest rate risk, which is correlated with the transition risk. This model is used later for the numerical experiments. It is assumed that the return R_n on firm n's assets until the risk horizon can be described by a normally distributed random variable, which is – without loss of generality – standardized:

$$R_n = \sqrt{\rho_R - \rho_{X_r,R}^2}\, Z + \rho_{X_r,R} X_r + \sqrt{1-\rho_R}\, \varepsilon_n \;\; (\rho_{X_r,R}^2 \le \rho_R,\; n \in \{1, \ldots, N\}) \tag{5}$$

where Z, X_r, and $\varepsilon_1, \ldots, \varepsilon_N$ are mutually independent, standard normally distributed stochastic variables. The variables Z and X_r represent systematic credit risk, by which all firms are affected, whereas ε_n stands for idiosyncratic credit risk. The specification (5) ensures that the correlation $Corr(R_n, R_m)$ between the asset returns of two different obligors n and m is equal to ρ_R. From the linear factor model (5) for the firms' asset returns, the conditional transition probabilities (see (1)) can be computed. The risk-free short rate is modeled, for simplicity, as a mean-reverting Ornstein-Uhlenbeck process introduced already by Vasicek (1977), which implies the following representation of the short rate at the risk horizon H:

$$r(H) = \theta + (r(0) - \theta)e^{-\kappa H} + \sqrt{\frac{\sigma_r^2}{2\kappa}\left(1 - e^{-2\kappa H}\right)}X_r \qquad (6)$$

where $\kappa, \theta, \sigma_r \in \mathbb{R}_+$. The random variable $X_r \sim N(0,1)$ also enters the definition (5) of the firms' asset returns. Thus, the correlation between the asset returns and the risk-free interest rates is $\rho_{X_r,R}$. The simulation of the possible ratings $1,\ldots,K$ of the obligors at the risk horizon proceeds exactly as described in the technical document of CreditMetrics (see Gupton et al. (1997, pp. 85)). The risk horizon H is set equal to one year.

Two different portfolio compositions are assumed: first, a portfolio of defaultable zero coupon bonds, and, second, a portfolio of European call options on default risk-free zero coupon bonds with counterparty risk. The price of a zero coupon bond at the risk horizon, whose issuer has not defaulted until this time, is calculated by discounting the future cash flow with the risk-adjusted spot yield appropriate for the issuer's rating at the risk horizon. This yield is composed of the stochastic risk-free spot yield (evolving according to the Vasicek model) and a non-stochastic credit spread of the respective rating class. In the Vasicek model the stochastic risk-free spot yield can easily be calculated in closed-form and is a linear function of the risk factor X_r appearing in (6). The credit spreads are not modeled as random variables to keep the integrated market and credit portfolio model as simple as possible. However, introducing stochastic credit spreads would be natural extension of the model considered in this paper (see Grundke (2005) for such a model). If an issuer of a zero coupon bond has already defaulted until the risk horizon, the value of the bond is set equal to a constant fraction δ of the value of a risk-free, but otherwise identical, zero coupon bond. The assumption of a constant recovery rate is again for simplicity; the IS technique would also work with (conditionally) independent recovery rates. For the pricing of an European call option on a (default) risk-free zero coupon bond with counterparty risk, it is assumed that a default is only possible at the maturity date T_n^C of the option. In this case, the recovery payment is an exogenous fraction δ of the option's regular pay off.[5] Furthermore, independence between the movements of the risk-free interest rates and the credit quality changes of the counterparties is assumed for the pricing of the options. With these assumptions, the price of a call written by counterparty n, whose rating at the risk horizon is $\eta_H^n \in \{1,\ldots,K-1\}$, is given by:

[5] This assumption can also be found, for example, in Klein (1996) and Klein and Inglis (2001). See the latter paper (pp. 997) also for an attempt to justify this, at first sight, rather restrictive assumption. Thus, for both instruments, the zero coupon bond as well as the interest rate option with counterparty risk, the creditor is assumed to get, in the case of a default of the obligor, a fraction δ of the value of a default risk-free but otherwise identical instrument at the default time.

$$C(\eta_H^n; X_r; \mathrm{P_n}) = \delta \tilde{C}(X_r; \mathrm{P_n}) + (1-\delta)\tilde{C}(X_r; \mathrm{P_n})\tilde{P}\left(\tau_n > T_n^C \middle| \eta_H^n\right). \tag{7}$$

Here, τ_n denotes the default time of counterparty n, $\tilde{P}$ is the risk-neutralized pricing measure, $\tilde{C}(X_r; \mathrm{P_n})$ is the price of an European call option *without* any counterparty risk in the term structure model of Vasicek (1977), and $\mathrm{P_n}$ is the vector of relevant option parameters (e.g., exercise price E_n, expiration date T_n^C). Assuming that a default is an absorbing state under $\tilde{P}$, the event $\{\tau_n > T_n^C\}$ is equivalent to the event $\{\eta_{T_n^C}^n \neq K\}$. The probability of the latter event can simply be calculated by summing up all individual risk-neutral probabilities for a rating change from η_H^n to a non-default state within the time interval $[H, T_n^C]$. Given the assumed independence between the risk-free interest rates and the rating transitions for pricing purposes, the transition probabilities under the risk-neutralized pricing measure $\tilde{P}$ can easily be inferred from the prices of defaultable bonds issued by the respective counterparty (see, e.g., Jarrow et al. (1997)).

4 Importance Sampling Techniques for the General Approach

When excess probabilities $P\left(L(H) > y\right)$ corresponding to large y are computed with ordinary Monte Carlo estimators

$$P\left(L(H) > y\right) = E^P\left[1_{\{L(H)>y\}}\right] \approx \frac{1}{D}\sum_{d=1}^{D} 1_{\{L(H)^{(d)}>y\}}, \tag{8}$$

a large number of simulation runs D is needed to achieve a sufficient accuracy. The basic idea of IS for estimating excess probabilities or percentiles, which are needed for example for VaR calculations, is to shift probability mass to the region of interest around y by changing the probability distribution of the underlying risk factors. The usage of IS can lead to an improved convergence (in probabilistic terms) of the risk measure estimators when increasing the number of simulation runs so that less simulation runs are necessary to achieve a required accuracy.

4.1 Application of a Two Step-Importance Sampling Technique

In this section, the two step-IS technique of GL (2005) is applied to the general integrated market and credit portfolio model described in section 3.1. If we changed only the probability distribution of the systematic risk factors Z and X, the resulting IS estimator of the excess probability $P\left(L(H) > y\right)$ is:

$$P\left(L(H) > y\right) \approx \frac{1}{D}\sum_{d=1}^{D} E^{P*}\left[1_{\{L(H)^{(d)}>y\}}\middle|\mathrm{Z}^{(d)},\mathrm{X}^{(d)}\right]\cdot\frac{f(\mathrm{Z}^{(d)},\mathrm{X}^{(d)})}{g(\mathrm{Z}^{(d)},\mathrm{X}^{(d)})}. \tag{9}$$

In (9), $\mathrm{Z}^{(d)}$ and $\mathrm{X}^{(d)}$ ($d \in \{1,\ldots,D\}$) are realizations of the systematic risk factors under the IS distribution $P*$ with density function $g(\mathrm{Z},\mathrm{X})$. $f(\mathrm{Z},\mathrm{X})$ is the density function of the systematic risk factors under the original distribution P, and the fraction, containing both density functions, in (9) is the so-called likelihood ratio, which reverses the transformation of the original probability measure. For computing the remaining conditional expectation in (9), the individual rating transitions *could* be sampled according to the original conditional transition probabilities (1). However, in a second step, these conditional transition probabilities are also modified to make larger losses more probable:

$$h_{n,i,k}(\mathrm{Z},\mathrm{X}) := \frac{e^{\theta L_{n,i,k}(\mathrm{X},H)} f_{n,i,k}(\mathrm{Z},\mathrm{X})}{E^{P}\left[e^{\theta L_{n,i,\eta_H^n}(\mathrm{X},H)}\middle|\mathrm{Z},\mathrm{X}\right]} \quad (1 \le i \le K-1; 1 \le k \le K) \tag{10}$$

where $L_{n,i,k}(\mathrm{X},H)$ is the loss of obligor n's instrument after a migration from rating grade i to k within the risk horizon, and θ is a non-negative parameter. For $\theta > 0$ and $L_{n,i,k}(\mathrm{X},H) > 0$, the transition probabilities are increased, whereas for $\theta > 0$ and $L_{n,i,k}(\mathrm{X},H) < 0$, they are diminished. The absolute increase or decrease of the probabilities is larger, the higher the individual losses or gains of obligor n's instrument incurred by the respective rating change are. For $\theta = 0$, the original transition probabilities are not altered. Hence, in general, the downgrade probabilities are increased and the upgrade probabilities are decreased. However, due to the integration of market risk, it is also possible that, for example, a downgrade probability is decreased, namely, in a specific scenario in which a decrease of an instrument's value caused by a downgrade of the issuer is overcompensated by a value increase caused by a movement in the market risk factors. The definition (10) of the conditional IS transition probabilities corresponds to a so-called exponential twist of the original conditional transition probabilities. This ensures that the conditional likelihood ratio has a nice representation: it only depends on the parameter θ, the simulated loss $L(H)$, and the cumulant generating function $\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta)$, which is the logarithm of the moment generating function of the portfolio loss variable $L(H)$. Thus, this yields the following representation of the conditional expectation in (9):

$$E^P\left[1_{\{L(H)>y\}}\middle|\mathrm{Z},\mathrm{X}\right]=E^{\tilde{P}_\theta}\left[1_{\{L(H)>y\}}e^{-\theta L(H)+\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta)}\middle|\mathrm{Z},\mathrm{X}\right] \tag{11}$$

where

$$\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta)=\ln\left(E^P\left[e^{\theta L(H)}\middle|\mathrm{Z},\mathrm{X}\right]\right). \tag{12}$$

Here, $E^{\tilde{P}_\theta}[\,\cdot\,]$ is the expectation operator under the IS probability measure $\tilde{P}_\theta$ for the conditional transitions, which depends on the parameter $\theta\in\mathbb{R}_+$.

The two essential problems which remain are finding an effective IS distribution for the systematic risk factors and for the conditional transition probabilities. Starting with the latter problem, the parameter θ has to be chosen in such a way that the variance or, equivalently, the second moment of the estimator under the IS distribution is minimized. As computing the second moment under the – up to now unknown – IS distribution is difficult, instead, an upper boundary of the second moment is minimized:

$$E^{\tilde{P}_\theta}\left[\left(1_{\{L(H)>y\}}e^{-\theta L(H)+\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta)}\right)^2\middle|\mathrm{Z},\mathrm{X}\right]\underset{\theta\geq 0}{\leq} e^{-2(\theta y-\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta))}. \tag{13}$$

Then, the in this sense optimal parameter θ is just given by the solution of the following optimization problem:

$$\theta_y(\mathrm{Z},\mathrm{X})=\begin{cases}\text{unique solution to } \dfrac{\partial}{\partial\theta}\left(\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta)\right)=y \text{ for } y>\dfrac{\partial}{\partial\theta}\left(\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta)\right)\Big|_{\theta=0}\\ 0 \qquad\qquad \text{for } y\leq\dfrac{\partial}{\partial\theta}\left(\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta)\right)\Big|_{\theta=0}\end{cases}. \tag{14}$$

Here, the θ's are restricted to non-negative values because otherwise the probability for transitions that yield a positive loss could be reduced. The twisting parameter $\theta_y(\mathrm{Z},\mathrm{X})$ is unique because the (conditional) cumulant generating function $\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta)$ is strictly convex in θ. Choosing the parameter $\theta_y(\mathrm{Z},\mathrm{X})$ according to (14) ensures that conditional credit portfolio losses around y are no longer rare events under the IS distribution $\tilde{P}_{\theta_y(\mathrm{Z},\mathrm{X})}$ for the conditional rating transitions. This can be seen from the relation (see Glasserman (2004, p. 261)):

$$E^{\tilde{P}_{\theta_y(\mathrm{Z},\mathrm{X})}}\left[L(H)\middle|\mathrm{Z},\mathrm{X}\right]=\frac{\partial}{\partial\theta}\left(\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta)\right)\Big|_{\theta=\theta_y(\mathrm{Z},\mathrm{X})}\underset{\substack{(14),\\ \text{upper case}}}{=}y. \tag{15}$$

Next, an IS density function for the systematic risk factors Z and X has to be found. In this situation, the optimal, this means zero-variance, IS density function for the systematic risk factors can be shown to be proportional to the product of the conditional probability and the original density function of the risk factors (see Glasserman (2004, p. 256)). However, sampling from this density is generally not possible because the normalization constant required to make it a density is just the probability $P(L(H)>y)$ that is looked for. For this problem, the following solution has been suggested: choose as IS distribution a normal density with the same mode as the optimal density, which is given by the solution of the following optimization problem:

$$\begin{aligned} \mu^{\mathrm{IS}}(y) &= (\mu_1^{\mathrm{Z}},\ldots,\mu_C^{\mathrm{Z}},\mu_1^{\mathrm{X}},\ldots,\mu_M^{\mathrm{X}})^T \\ &= \underset{z_1,\ldots,z_C,x_1,\ldots,x_M\in\mathbb{R}}{\arg\max}\ E^{\tilde{P}_{\theta_y(\mathrm{z},\mathrm{x})}}\left[1_{\{L(H)>y\}}e^{-\theta_y(\mathrm{z},\mathrm{x})L(H)+\psi_{L(H)|\mathrm{Z}=\mathrm{z},\mathrm{X}=\mathrm{x}}(\theta_y(\mathrm{z},\mathrm{x}))}\middle|\mathrm{Z}=\mathrm{z},\mathrm{X}=\mathrm{x}\right]\cdot e^{-0.5\left(\sum_{c=1}^{C}z_c^2+\sum_{m=1}^{M}x_m^2\right)} \end{aligned} \qquad (16)$$

In (16), the IS representation for the conditional excess probability $P\left(L(H)>y\middle|\mathrm{Z},\mathrm{X}\right)$ has been inserted. Furthermore, it has been assumed that the systematic risk factors are uncorrelated, standard normally distributed random variables under the original probability distribution P.[6] To simplify the above optimization problem, again an upper boundary of the conditional expectation is used, which yields the related optimization problem:

$$\mu^{\mathrm{IS}}(y) \approx \underset{z_1,\ldots,z_C,x_1,\ldots,x_M\in\mathbb{R}}{\arg\max}\ -\theta_y(\mathrm{z},\mathrm{x})y+\psi_{L(H)|\mathrm{Z}=\mathrm{z},\mathrm{X}=\mathrm{x}}(\theta_y(\mathrm{z},\mathrm{x}))-0.5\left(\sum_{c=1}^{C}z_c^2+\sum_{m=1}^{M}x_m^2\right). \qquad (17)$$

The IS distribution for the systematic risk factors Z and X is a multivariate normal distribution with mean vector given by (17) and covariance matrix equal to the identity matrix I, which is also the covariance matrix of the systematic risk factors under the original probability distribution P. In the end, this yields the following two step-IS estimator for the excess probabilities:

$$P(L(H)>y) \approx \frac{1}{D}\sum_{d=1}^{D}1_{\{L(H)^{(d)}>y\}}\cdot l_1^y(\mathrm{Z}^{(d)},\mathrm{X}^{(d)})\cdot l_2(\mathrm{Z}^{(d)},\mathrm{X}^{(d)}) \qquad (18)$$

where

$$l_1^y(\mathrm{Z}^{(d)},\mathrm{X}^{(d)}) := e^{-\theta_y(\mathrm{Z}^{(d)},\mathrm{X}^{(d)})L(H)^{(d)}+\psi_{L(H)|\mathrm{Z}^{(d)},\mathrm{X}^{(d)}}(\theta_y(\mathrm{Z}^{(d)},\mathrm{X}^{(d)}))} \qquad (19)$$

and

$$l_2(\mathrm{Z}^{(d)},\mathrm{X}^{(d)}) := e^{-\sum_{c=1}^{C}\left(Z_c^{(d)}\mu_c^{\mathrm{Z}}-0.5(\mu_c^{\mathrm{Z}})^2\right)-\sum_{m=1}^{M}\left(X_m^{(d)}\mu_m^{\mathrm{X}}-0.5(\mu_m^{\mathrm{X}})^2\right)}. \qquad (20)$$

Here, the first exponential term is the likelihood ratio for the conditional IS transition probabilities, and the second exponential term is the likelihood ratio for the IS distribution of the systematic risk factors. The systematic risk factors are sampled according to

6 If the joint distribution of the random vector (Z,X) is a multivariate normal distribution, this assumption is without loss of generality, because a set of correlated normally distributed random variables can always be represented by a linear combination of orthogonal standard normally distributed random variables.

$N((\mu_1^Z,\ldots,\mu_C^Z,\mu_1^X,\ldots,\mu_M^X)^T,\mathrm{I})$ and – conditional on (Z,X) – the portfolio loss $L(H)$ is sampled according to the IS transition probabilities $h_{n,i,k}(\mathrm{Z},\mathrm{X})$ (see (10)).

4.2 Discussion of the Two Step-Importance Sampling Technique

GL (2005) apply the above two step-IS procedure to a CreditMetrics-style, pure default mode model. In their modeling approach, the (conditional) cumulant generating function and its derivative with respect to θ, respectively, are given by:

$$\psi_{L(H)|\mathrm{Z}}(\theta)=\sum_{n=1}^{N}\ln\left(1+f_{n,\eta_0^n,K}(\mathrm{Z})\left(e^{\theta c_n}-1\right)\right) \tag{21}$$

and

$$\frac{\partial}{\partial\theta}\left(\psi_{L(H)|\mathrm{Z}}(\theta)\right)=\sum_{n=1}^{N}\frac{f_{n,\eta_0^n,K}(\mathrm{Z})c_n e^{\theta c_n}}{1+f_{n,\eta_0^n,K}(\mathrm{Z})\left(e^{\theta c_n}-1\right)}. \tag{22}$$

Here, $f_{n,\eta_0^n,K}(\mathrm{Z})$ is the conditional default probability of obligor n, and c_n is the loss incurred by a default of this obligor. As

$$\left.\frac{\partial}{\partial\theta}\left(\psi_{L(H)|\mathrm{Z}}(\theta)\right)\right|_{\theta=0}=E^P\left[L(H)\middle|\mathrm{Z}\right] \tag{23}$$

and

$$\lim_{\theta\to\infty}\frac{\partial}{\partial\theta}\left(\psi_{L(H)|\mathrm{Z}}(\theta)\right)=\sum_{n=1}^{N}c_n, \tag{24}$$

the optimization problem (14) has a solution $\theta\in(0,\infty)$ for all values y larger than the conditional expected loss (23) under the original transition probabilities and smaller than the maximum loss (24).

In the extended market and credit portfolio model considered in this paper, the potential loss is itself a random variable, which depends on the realization of the market risk factors. Defining the conditional loss of obligor n's instrument, given that the obligor's rating at the risk horizon is k, as

$$L_{n,k}(H)\middle|\mathrm{X}:=p_n(\eta_0^n;E^P[\mathrm{X}];\mathrm{P_n})-p_n(k;\mathrm{X};\mathrm{P_n}), \tag{25}$$

the optimization problem (14) has a solution $\theta\in(0,\infty)$ for all values

$$y\in\left(\underbrace{\sum_{n=1}^{N}E^P\left[L_{n,\eta_H^n}(H)\middle|\mathrm{Z},\mathrm{X}\right]}_{\text{conditional expected loss}},\underbrace{\sum_{n=1}^{N}\max_{1\le k\le K}L_{n,k}(H)\middle|\mathrm{X}}_{\text{conditional maximum loss}}\right). \tag{26}$$

As

$$\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta)=\sum_{n=1}^{N}\ln\left(\sum_{k=1}^{K}e^{\theta L_{n,k}(H)|\mathrm{X}}f_{n,\eta_0^n,k}(\mathrm{Z})\right) \tag{27}$$

and hence

$$\frac{\partial}{\partial\theta}\left(\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta)\right)=\sum_{n=1}^{N}\frac{\sum_{k=1}^{K}e^{\theta L_{n,k}(H)|\mathrm{X}}f_{n,\eta_0^n,k}(\mathrm{Z})\,L_{n,k}(H)\big|\mathrm{X}}{\sum_{k=1}^{K}e^{\theta L_{n,k}(H)|\mathrm{X}}f_{n,\eta_0^n,k}(\mathrm{Z})}, \tag{28}$$

the lower boundary in (26) follows directly from setting $\theta=0$ in (28). Without loss of generality, $L_{n,K}(H)\big|\mathrm{X}=\max_{1\le k\le K}L_{n,k}(H)\big|\mathrm{X}$ can be assumed for all $n\in\{1,\ldots,N\}$. This assumption implies that the maximum loss is always incurred by a default. Then, the upper boundary in (26) can be seen from:

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta)\right)&=\sum_{n=1}^{N}\left(\frac{\sum_{k=1}^{K}e^{\theta L_{n,k}(H)|\mathrm{X}}f_{n,\eta_0^n,k}(\mathrm{Z})\,L_{n,k}(H)\big|\mathrm{X}}{\sum_{k=1}^{K}e^{\theta L_{n,k}(H)|\mathrm{X}}f_{n,\eta_0^n,k}(\mathrm{Z})}\cdot\frac{e^{-\theta L_{n,K}(H)|\mathrm{X}}}{e^{-\theta L_{n,K}(H)|\mathrm{X}}}\right)\\&=\sum_{n=1}^{N}\left(\frac{\sum_{k=1}^{K-1}e^{\theta\left(L_{n,k}(H)|\mathrm{X}-L_{n,K}(H)|\mathrm{X}\right)}f_{n,\eta_0^n,k}(\mathrm{Z})\,L_{n,k}(H)\big|\mathrm{X}+f_{n,\eta_0^n,K}(\mathrm{Z})\,L_{n,K}(H)\big|\mathrm{X}}{\sum_{k=1}^{K-1}e^{\theta\left(L_{n,k}(H)|\mathrm{X}-L_{n,K}(H)|\mathrm{X}\right)}f_{n,\eta_0^n,k}(\mathrm{Z})+f_{n,\eta_0^n,K}(\mathrm{Z})}\right).\end{aligned} \tag{29}$$

Due to $L_{n,k}(H)\big|\mathrm{X}-L_{n,K}(H)\big|\mathrm{X}<0$ for all $k\in\{1,\ldots,K-1\}$, (29) converges to the upper boundary in (26) for $\theta\to\infty$.

However, depending on the instrument type and the realization of the market risk factors X, the interval (26) can be rather small. This problem occurs when losses are mainly caused by changes in the market risk factors and not by rating transitions. This is true, for example, for options portfolios, which are considered in the numerical experiments. The consequence is that the optimization problem (14) might not be well defined. If $y>\sum_{n=1}^{N}\max_{1\le k\le K}L_{n,k}(H)\big|\mathrm{X}^{(d)}$ for a specific simulation run $(\mathrm{Z}^{(d)},\mathrm{X}^{(d)})$, this yields $\theta=\infty$ in (14). In this case, the likelihood ratio of the conditional transition probabilities converges with probability one to the product of the conditional default probabilities:

$$\exp\left(-\theta_y(\mathrm{Z}^{(d)},\mathrm{X}^{(d)})L(H)^{(d)}+\psi_{L(H)|\mathrm{Z}^{(d)},\mathrm{X}^{(d)}}(\theta_y(\mathrm{Z}^{(d)},\mathrm{X}^{(d)}))\right)\underset{\theta_y(\mathrm{Z}^{(d)},\mathrm{X}^{(d)})\to\infty}{\longrightarrow}\prod_{n=1}^{N}f_{n,\eta_0^n,K}(\mathrm{Z}^{(d)}). \tag{30}$$

Here, $L_{n,K}(H)\big|\mathrm{X} = \max_{1\le k\le K} L_{n,k}(H)\big|\mathrm{X}$ is again assumed for all $n \in \{1,\dots,N\}$. The limit result (30) can be seen as follows. For the conditional likelihood ratio, the following representation can be derived:

$$\exp\Big(-\theta_y(\mathrm{Z}^{(d)},\mathrm{X}^{(d)})L(H)^{(d)} + \psi_{L(H)|\mathrm{Z}^{(d)},\mathrm{X}^{(d)}}(\theta_y(\mathrm{Z}^{(d)},\mathrm{X}^{(d)}))\Big)$$

$$= \exp\left(\sum_{n=1}^{N}\Big(-\theta_y(\mathrm{Z}^{(d)},\mathrm{X}^{(d)})\Big)L_{n,\eta_H^n}(H)\Big|\mathrm{X}^{(d)} + \sum_{n=1}^{N}\ln\left(\sum_{k=1}^{K} e^{\theta_y(\mathrm{Z}^{(d)},\mathrm{X}^{(d)})L_{n,k}(H)|\mathrm{X}^{(d)}} f_{n,\eta_0^n,k}(\mathrm{Z}^{(d)})\right)\right)$$

$$= \prod_{n=1}^{N}\exp\left(\Big(-\theta_y(\mathrm{Z}^{(d)},\mathrm{X}^{(d)})\Big)L_{n,\eta_H^n}(H)\Big|\mathrm{X}^{(d)} + \ln\left(\sum_{k=1}^{K} e^{\theta_y(\mathrm{Z}^{(d)},\mathrm{X}^{(d)})L_{n,k}(H)|\mathrm{X}^{(d)}} f_{n,\eta_0^n,k}(\mathrm{Z}^{(d)})\right)\right)$$

$$= \prod_{n=1}^{N}\left(\sum_{k=1}^{K}\exp\Big(\theta_y(\mathrm{Z}^{(d)},\mathrm{X}^{(d)})\Big(L_{n,k}(H)\Big|\mathrm{X}^{(d)} - L_{n,\eta_H^n}(H)\Big|\mathrm{X}^{(d)}\Big)\Big)f_{n,\eta_0^n,k}(\mathrm{Z}^{(d)})\right). \tag{31}$$

From the definition (10) of the conditional transition probabilities under the IS probability measure $\tilde{P}_{\theta_y(\mathrm{Z}^{(d)},\mathrm{X}^{(d)})}$ follows that for $\theta_y(\mathrm{Z}^{(d)},\mathrm{X}^{(d)}) \to \infty$ all non-default probabilities converge to zero, whereas the default probability goes to one:

$$h_{n,\eta_0^n,k}(\mathrm{Z}^{(d)},\mathrm{X}^{(d)}) = \frac{e^{\theta_y(\mathrm{Z}^{(d)},\mathrm{X}^{(d)})L_{n,k}(H)|\mathrm{X}^{(d)}} f_{n,\eta_0^n,k}(\mathrm{Z}^{(d)})}{\sum_{s=1}^{K} e^{\theta_y(\mathrm{Z}^{(d)},\mathrm{X}^{(d)})L_{n,s}(H)|\mathrm{X}^{(d)}} f_{n,\eta_0^n,s}(\mathrm{Z}^{(d)})} \cdot \frac{e^{-\theta_y(\mathrm{Z}^{(d)},\mathrm{X}^{(d)})L_{n,K}(H)|\mathrm{X}^{(d)}}}{e^{-\theta_y(\mathrm{Z}^{(d)},\mathrm{X}^{(d)})L_{n,K}(H)|\mathrm{X}^{(d)}}}$$

$$= \frac{e^{\theta_y(\mathrm{Z}^{(d)},\mathrm{X}^{(d)})\left(L_{n,k}(H)|\mathrm{X}^{(d)} - L_{n,K}(H)|\mathrm{X}^{(d)}\right)} f_{n,\eta_0^n,k}(\mathrm{Z}^{(d)})}{\sum_{s=1}^{K-1} e^{\theta_y(\mathrm{Z}^{(d)},\mathrm{X}^{(d)})\left(L_{n,s}(H)|\mathrm{X}^{(d)} - L_{n,K}(H)|\mathrm{X}^{(d)}\right)} f_{n,\eta_0^n,s}(\mathrm{Z}^{(d)}) + f_{n,\eta_0^n,K}(\mathrm{Z}^{(d)})}$$

$$\underset{\theta_y(\mathrm{Z}^{(d)},\mathrm{X}^{(d)})\to\infty}{\longrightarrow} \begin{cases} 0 & \text{for } k \in \{1,\dots,K-1\}, \\ 1 & \text{for } k = K. \end{cases}$$

Thus, in the limit, this yields $L_{n,\eta_H^n}(H)\big|\mathrm{X}^{(d)} = L_{n,K}(H)\big|\mathrm{X}^{(d)}$ with probability one in (31). This implies $L_{n,k}(H)\big|\mathrm{X}^{(d)} - L_{n,\eta_H^n}(H)\big|\mathrm{X}^{(d)} < 0$ for all $k \in \{1,\dots,K-1\}$ and $L_{n,K}(H)\big|\mathrm{X}^{(d)}$ $-L_{n,\eta_H^n}(H)\big|\mathrm{X}^{(d)} = 0$ with probability one, from which (30) results. As the default probabilities $f_{n,\eta_0^n,K}(\mathrm{Z}^{(d)})$ are usually very small, (30) will already be indistinguishable from zero for the computer for a moderate number N of obligors. The consequence is that this simulation run has no contribution to the probability estimator (18). In the CreditMetrics-style, pure default mode model, which GL (2005) employ for their numerical examples, this problem can not occur because no initial percentile guess y should be larger than the non-stochastic maximum potential loss $\sum_{n=1}^{N} c_n$.

If $y < \sum_{n=1}^{N} E^P \left[L_{n,\eta_H^n}(H) \middle| \mathrm{Z},\mathrm{X} \right]$, the parameter θ is set equal to zero (see (14)). This just implies that there is no IS for the conditional transition probabilities and that the conditional likelihood ratio is one. However, even if y is in the interval (26), the parameter θ fulfilling $\frac{\partial}{\partial\theta}\left(\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta)\right) = y$ can be very large (see the example in section 5.3 in the following). This can cause an overflow problem when calculating the conditional cumulant generating function and its derivative.

4.3 Modification: Insertion of a Third Step

As the topic of this paper are integrated market and credit portfolio model, it might suggest itself to employ also IS techniques originally developed for pure market risk portfolio models and to combine these with those techniques originally developed for pure credit risk portfolio models. This is tried next. As a result, a three step-IS algorithm is developed. Of course, the hypothesis is that this three step-IS technique is even more effective than the two step-IS approach described in the previous section.

Instead of considering the optimization problem (17) for finding the IS means of both kinds of systematic risk factors, now, this procedure is only carried out for the systematic credit risk factors $Z_1, \ldots, Z_C$, whereas the IS distribution for the market risk factors $X_1, \ldots, X_M$ is determined in an intermediate step. For finding the IS distribution of the market risk factors, it is assumed that the credit portfolio is default risk-free and that all obligors remain in their initial rating class until the risk horizon. With this assumption, the approach of Glasserman et al. (2000) developed for pure market risk portfolio models can be applied. Their method employs a delta-gamma approximation of the portfolio loss at the risk horizon for selecting a variance reducing IS distribution of the market risk factors. Here, as in the previous section, it is assumed that the market risk factors are uncorrelated and standard normally distributed.[7] The random variable representing the credit portfolio loss, which is only due to movements in the market risk factors over the risk horizon, is defined as:

7 For the necessary transformations when the market risk factors are multivariate normally distributed (in particular correlated), see Glasserman (2004, pp. 486) and Glasserman et al. (2000, p. 1351). For lognormally or t-distributed risk factors, see Glasserman et al. (2000, p. 1351)) and Glasserman et al. (2002).

$$L^{wtr}(\mathrm{X},H)=\sum_{n=1}^{N}L_n^{wtr}(\mathrm{X},H)=\sum_{n=1}^{N}\left(p_n(\eta_0^n;E^P[\mathrm{X}];\mathrm{P_n})-p_n(\eta_0^n;\mathrm{X};\mathrm{P_n})\right) \tag{32}$$

where the upper index wtr indicates that this is the loss *w*ithout *t*ransition *r*isk. Note that both terms in the sum in (32) depend on the current rating η_0^n. For this random variable, a specific quadratic approximation, the so-called delta-gamma approximation, is introduced:

$$\begin{aligned} &L^{wtr}(\mathrm{X},H)\approx L^{wtr,\Delta,\Gamma}(\mathrm{X},H) \\ &=\underbrace{L^{wtr}(E^P[\mathrm{X}],H)}_{=0}+\delta^{\mathrm{T}}(\mathrm{X}-\underbrace{E^P[\mathrm{X}]}_{=0})+0.5(\mathrm{X}-\underbrace{E^P[\mathrm{X}]}_{=0})^T\Gamma(\mathrm{X}-\underbrace{E^P[\mathrm{X}]}_{=0}) \end{aligned} \tag{33}$$

where the column vector $\delta=(\delta_m)_{1\le m\le M}$ contains the first derivatives of $L^{wtr}(\mathrm{X},H)$ with respect to the market risk factors, and the matrix $\Gamma=(\Gamma_{m,l})_{1\le m,l\le M}$ is the Hessian matrix with the second derivatives of $L^{wtr}(\mathrm{X},H)$ with respect to the market risk factors. Hence, (33) is just a second-order Taylor series expansion of the credit portfolio loss $L^{wtr}(\mathrm{X},H)$ around the expected market risk factors at the risk horizon. After diagonalization of the Hessian matrix Γ, the following more convenient expression for the quadratic approximation (33) can be derived:

$$L^{wtr}(\mathrm{X},H)\approx L^{wtr,\Delta,\Gamma}(\mathrm{X},H)=\sum_{m=1}^{M}\left(b_mS_m+\lambda_mS_m^2\right)=Q(\mathrm{S}) \tag{34}$$

where $\mathrm{S}=\mathrm{U}^T\mathrm{X}\sim N(0,\mathrm{I})$, U is an orthogonal matrix whose columns are the eigenvectors of 0.5Γ, λ_m ($m\in\{1,\ldots,M\}$) are the components of the diagonal matrix Λ containing the eigenvalues of $0.5\Gamma=\mathrm{U}\Lambda\mathrm{U}^T$, and b_m ($m\in\{1,\ldots,M\}$) are the components of the vector $\delta^T\mathrm{U}$. In the next step, the approximation $L^{wtr}(\mathrm{X},H)\approx Q(\mathrm{S})$ is used for finding an IS distribution for the transformed market risk factors S under which large values of the portfolio loss $L^{wtr}(\mathrm{X},H)$ are generated with an higher probability than under the original distribution. As IS distribution, a multivariate normal distribution with means

$$\mu_m(\theta^{wtr})=\frac{\theta^{wtr}b_m}{1-2\theta^{wtr}\lambda_m} \quad (m\in\{1,\ldots,M\}), \tag{35}$$

and covariances

$$\sigma_{ml}^2(\theta^{wtr})=\begin{cases}\dfrac{1}{1-2\theta^{wtr}\lambda_m} & (m=l)\\ 0 & (m\neq l)\end{cases} \quad (m,l\in\{1,\ldots,M\}) \tag{36}$$

is chosen. For the parameter θ^{wtr}, it is required that $0\le\theta^{wtr}<(\max_{1\le m\le M}2\lambda_m)^{-1}$ if $\max_{1\le m\le M}\lambda_m>0$. Choosing the IS means and variances as defined in (35) and (36) ensures that those random variables S_m for which $b_m > 0$ ($b_m < 0$) have a positive (negative) mean and that those S_m for which $\lambda_m > 0$ have a large variance. This in turn makes larger values of $Q(\mathrm{S})$ and hence larger portfolio losses $L^{wtr}(\mathrm{X}, H)$ more likely. A further consequence of the choice (35) and (36) is the following simple likelihood ratio, which is typical for exponential twisting:

$$l_3(\mathrm{S}) = \frac{e^{-\frac{1}{2}\mathrm{S}^{\mathrm{T}}\mathrm{S}}}{\det(\Sigma(\theta^{wtr}))^{-\frac{1}{2}} e^{-\frac{1}{2}(\mathrm{S}-\mu(\theta^{wtr}))^{\mathrm{T}}\Sigma(\theta^{wtr})^{-1}(\mathrm{S}-\mu(\theta^{wtr}))}} = e^{-\theta^{wtr}Q(\mathrm{S})+\psi_Q(\theta^{wtr})} \tag{37}$$

where $\psi_Q(\theta^{wtr}) = \ln\left(E^P\left[e^{\theta^{wtr}Q(\mathrm{S})}\right]\right)$ is the cumulant generating function of the random variable $Q(\mathrm{S})$. In a final step, the twisting parameter θ^{wtr} has to be determined. For this, the approximation $P(L^{wtr}(H) > y^*) \approx P(Q(\mathrm{S}) > y^*)$ is used and a parameter θ^{wtr}, which is effective for estimating the probability on the right-hand side, is computed, hoping that it is also effective for estimating the probability on the left-hand side.[8] Again, the parameter θ^{wtr} is computed by minimizing an upper boundary of the second moment of the excess probability estimator under the IS distribution:

$$E^{\tilde{P}_{\theta^{wtr}}}\left[\left(1_{\{Q(\mathrm{S})>y^*\}} l_3(\mathrm{S})\right)^2\right] \underset{\theta\geq 0}{\leq} e^{-2\left(\theta^{wtr}y^* - \psi_Q(\theta^{wtr})\right)}. \tag{38}$$

This yields:

$$\theta_{y^*}^{wtr} = \begin{cases} \text{unique solution to } \dfrac{\partial}{\partial\theta^{wtr}}\left(\psi_Q(\theta^{wtr})\right) = y^* & \text{for } y^* > \left.\dfrac{\partial}{\partial\theta^{wtr}}\left(\psi_Q(\theta^{wtr})\right)\right|_{\theta^{wtr}=0}, \\ 0 & \text{for } y^* \leq \left.\dfrac{\partial}{\partial\theta^{wtr}}\left(\psi_Q(\theta^{wtr})\right)\right|_{\theta^{wtr}=0}. \end{cases} \tag{39}$$

As

$$\left.\frac{\partial}{\partial\theta^{wtr}}\left(\psi_Q(\theta^{wtr})\right)\right|_{\theta^{wtr}=0} = \sum_{m=1}^{M}\lambda_m \tag{40}$$

and

$$\frac{\partial}{\partial\theta^{wtr}}\left(\psi_Q(\theta^{wtr})\right) \underset{\theta^{wtr}\to\infty}{\to} -\sum_{m=1}^{M}\frac{b_m^2}{4\lambda_m}, \tag{41}$$

the above optimization problem (39) is well defined and has a non-zero solution for initial guesses y^* out of the interval

[8] In the following, it is differed between the two figures y and y^*: y is the initial guess of the percentile of the credit portfolio loss distribution which is looked for, whereas y^* is the initial guess of a percentile of the loss distribution when only market risk, but no transition risk, is considered. Especially for portfolios with a low credit quality, these figures differ significantly, even if the percentiles correspond to the same confidence level.

$$y^* \in \left(\sum_{m=1}^{M} \lambda_m, -\sum_{m=1}^{M} \frac{b_m^2}{4\lambda_m} \right). \tag{42}$$

However, due to the additional restriction for θ^{wtr} mentioned after (36), which ensures that the variance of the market risk factors under the IS distribution is non-negative, the admissible interval for y^* might even have a smaller upper boundary.

4.4 Combination of the Three Steps

Next, the problem arises how to combine the two step-IS technique for the systematic credit risk factors Z with the IS technique developed for pure market risk portfolio models to build up a three step-IS technique. There are several possibilities to do this, which are discussed in the following. As the excess probability $P(L(H) > y)$ can be represented by:

$$\begin{aligned}
&P(L(H) > y) = E^P\left[P\left(L(H) > y \middle| \mathrm{Z}, \mathrm{X}\right)\right] \\
&= E^P\left[E^{\tilde{P}_{\theta_y(\mathrm{Z},\mathrm{US})}}\left[1_{\{L(H)>y\}} l_1(Z, \mathrm{US}) \middle| \mathrm{Z}, \mathrm{S}\right]\right] \\
&= E^P\left[E^P\left[E^{\tilde{P}_{\theta_y(\mathrm{Z},\mathrm{US})}}\left[1_{\{L(H)>y\}} l_1(\mathrm{Z}, \mathrm{US}) \middle| \mathrm{Z}, \mathrm{S}\right] \middle| \mathrm{Z}\right]\right] \\
&= E^P\left[E^{\tilde{P}_{\theta_{y^*}^{wtr}}}\left[E^{\tilde{P}_{\theta_y(\mathrm{Z},\mathrm{US})}}\left[1_{\{L(H)>y\}} l_1(\mathrm{Z}, \mathrm{US}) \middle| \mathrm{Z}, \mathrm{S}\right] l_3(\mathrm{S}) \middle| \mathrm{Z}\right]\right],
\end{aligned}$$

this yields the following three step-IS estimator for the excess probability $P(L(H) > y)$:

$$P(L(H) > y) \approx \frac{1}{D} \sum_{d=1}^{D} 1_{\{L(H)^{(d)}>y\}} \cdot l_1(\mathrm{Z}^{(d)}, \mathrm{US}^{(d)}) \cdot l_3(\mathrm{S}^{(d)}) \cdot l_2(\mathrm{Z}^{(d)}). \tag{43}$$

Here, the likelihood ratio $l_1(\mathrm{Z}^{(d)}, \mathrm{US}^{(d)})$ for the conditional transition probabilities is defined as in (19), the likelihood ratio $l_3(\mathrm{S}^{(d)})$ for the market risk factors S is given by (37), and the likelihood ratio $l_2(\mathrm{Z}^{(d)})$ for the systematic credit risk factors Z is defined analogously to (20). Conditional on the realization of (Z, S), the credit portfolio loss $L(H)$ is sampled according to the IS transition probabilities $h_{n,i,k}(\mathrm{Z}, \mathrm{US})$, the vector of transformed market risk factors S is sampled according to (35) and (36), and the vector of systematic credit risk factors Z is sampled according to $N(\mu^{\mathrm{IS}}(y), \mathrm{I})$. Employing an upper boundary for $E^{\tilde{P}_{\theta_y(\mathrm{Z},\mathrm{US})}}\left[1_{\{L(H)>y\}} l_1(\mathrm{Z}, \mathrm{US}) \middle| \mathrm{Z}, \mathrm{S}\right]$ analogously to (13), the C-dimensional IS mean vector $\mu^{\mathrm{IS}}(y)$ for the systematic credit risk factors Z is given by:

$$\mu^{\mathrm{IS}}(y)=\underset{z_1,\ldots,z_C\in\mathbb{R}}{\arg\max}\, E^{\tilde{P}_{\theta^{wtr}_{y*}}}\left[e^{-\theta_y(\mathrm{z},\mathrm{US})y+\psi_{L(H)|\mathrm{z},\mathrm{US}}(\theta_y(\mathrm{z},\mathrm{US}))}l_3(\mathrm{S})\middle|\mathrm{Z}=\mathrm{z}\right]e^{-0.5\sum_{c=1}^{C}z_c^2}. \tag{44}$$

Solving the optimization problem (44) would be rather involved because there are usually many market risk factors S, which are relevant for the value of a portfolio, and hence a multi-dimensional integral would have to be solved numerically many times in this optimization problem. To circumvent this drawback, an alternative might be to change the order in which the conditional expectations are computed. This yields:

$$\begin{aligned}
&P(L(H)>y)\\
&=E^P\left[P\left(L(H)>y\middle|\mathrm{Z},\mathrm{X}\right)\right]\\
&=E^P\left[E^P\left[E^{\tilde{P}_{\theta_y(\mathrm{Z},\mathrm{US})}}\left[1_{\{L(H)>y\}}l_1(\mathrm{Z},\mathrm{US})\middle|\mathrm{Z},\mathrm{S}\right]\middle|\mathrm{S}\right]\right]\\
&=E^{\tilde{P}_{\theta^{wtr}_{y*}}}\left[E^P\left[E^{\tilde{P}_{\theta_y(\mathrm{Z},\mathrm{US})}}\left[1_{\{L(H)>y\}}l_1(\mathrm{Z},\mathrm{US})\middle|\mathrm{Z},\mathrm{S}\right]\middle|\mathrm{S}\right]l_3(\mathrm{S}^{(d)})\right]\\
&=E^{\tilde{P}_{\theta^{wtr}_{y*}}}\left[E^{P_{\mu^{\mathrm{IS}}_y(\mathrm{S})}}\left[E^{\tilde{P}_{\theta_y(\mathrm{Z},\mathrm{US})}}\left[1_{\{L(H)>y\}}l_1(\mathrm{Z},\mathrm{US})\middle|\mathrm{Z},\mathrm{S}\right]l_2(\mathrm{Z})|\mathrm{S}\middle|\mathrm{S}\right]l_3(\mathrm{S}^{(d)})\right]
\end{aligned}$$

where the likelihood ratio

$$l_2(\mathrm{Z})|\mathrm{S}=e^{-\sum_{c=1}^{C}\left(Z_c\mu_c^{\mathrm{Z}}(\mathrm{S})-0.5(\mu_c^{\mathrm{Z}}(\mathrm{S}))^2\right)} \tag{45}$$

for the systematic risk factors Z as well as their IS means $\mu^{\mathrm{IS}}_y(\mathrm{S})=(\mu_1^{\mathrm{Z}}(\mathrm{S}),\ldots,\mu_C^{\mathrm{Z}}(\mathrm{S}))^T$

$$\mu^{\mathrm{IS}}_y(\mathrm{S})=\underset{z_1,\ldots,z_C\in\mathbb{R}}{\arg\max}\left(-\theta_y(\mathrm{z},\mathrm{US})y+\psi_{L(H)|\mathrm{z},\mathrm{US}}(\theta_y(\mathrm{z},\mathrm{US}))-0.5\sum_{c=1}^{C}z_c^2\right) \tag{46}$$

now depend on the realization of the market risk factors S. Unfortunately, this optimization problem has the serious disadvantage that it has to be solved for each scenario of the market risk factors S, which makes this approach also computational expensive. A third possibility to combine all three steps, which avoids the computational difficulties of the two previous approaches, is to assume that the market risk factors S equal their IS means $E^{\tilde{P}_{\theta^{wtr}_{y*}}}[S]=\mu(\theta^{wtr}_{y*})$ while determining the IS means of the systematic credit risk factors Z. This yields the following optimization problem (instead of (44) and (46)):

$$\mu^{\mathrm{IS}}_y=\underset{z_1,\ldots,z_C\in\mathbb{R}}{\arg\max}\left(-\theta_y(\mathrm{z},\mathrm{U}\mu(\theta^{wtr}_{y*}))y+\psi_{L(H)|\mathrm{z},\mathrm{U}\mu(\theta^{wtr}_{y*})}(\theta_y(\mathrm{z},\mathrm{U}\mu(\theta^{wtr}_{y*})))-0.5\sum_{c=1}^{C}z_c^2\right). \tag{47}$$

The effectiveness of this simplified third approach is tested within the numerical example in the next section.

5 Numerical Results

Next, the effectiveness of the IS techniques presented in the previous sections is analyzed by means of numerical experiments. For this, the simple example of an integrated market and credit portfolio model as described in section 3.2 is employed. For a one-factor CreditMetrics-style, default mode model applied to a homogeneous portfolio, GL (2005) are able to derive various theoretical results (mainly concerning asymptotic optimality) about the two step-IS technique they propose. However, even for the simple integrated market and credit portfolio model which is employed in this section, it seems rather difficult to derive similar results. That is why only numerical experiments, which are intended to demonstrate the performance of the different IS techniques, are carried out.

5.1 Parameters

First, it is assumed that the credit portfolio consists of $N = 500$ defaultable zero coupon bonds, which are issued by N different obligors, but are otherwise identical. The face value is chosen to be $F = 1$ and the maturity date is $T = 3$, implying a remaining time to maturity of two years at the risk horizon. The simulations are done for the initial ratings $\eta_0 \in \{\text{Aa, Baa, B}\}$. As typical parameters for the Vasicek term structure model, $\kappa = 0.4$ and $\sigma_r = 0.01$ are chosen. The mean level θ and the initial short rate $r(0)$ are set equal to 0.06. As market price of interest rate risk λ, a value of 0.5 is taken.[9] The recovery rate is set equal to 53.80%, which is the mean of the recovery rate of senior unsecured bonds during 1970 to 1995 reported by Moody's.[10] The employed transition matrix is also from Moody's.[11] The value of the correlation parameter ρ_R of the asset returns is chosen as 0.1 and 0.4, respectively. The former value is within the range of values proposed by the Basle Committee on Banking Supervision for corporate exposures in the IRB approach (see Basel Committee on Banking Supervision (2005)). The latter value is taken to test the effect of extreme asset re-

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9 For example, Barnhill and Maxwell (2002) estimate a short rate volatility of 0.007, whereas Lehrbass (1997) finds $\sigma_r = 0.029$, and Huang and Huang (2003) even work with $\sigma_r = 0.0468$. With regard to the mean reversion parameter and the market price of interest rate risk, Lehrbass finds $\kappa = 1.169$ and absolute values of 0.59, 0.808 and 1.232 for the parameter λ, whereas Huang and Huang choose $\kappa = 0.226$ and an absolute value of 0.248 for λ.

10 See Moody's Investors Service (1996). However, using Moody's estimate of the mean ignores the fact that the rating agency defines the recovery rate as a percentage of par and not as a percentage of a risk-free but otherwise identical zero coupon bond.

11 See Moody's Investors Service (2002, p. 31). The probabilities are average values of all corporates in the period 1970-2001. The category 'rating withdrawn' is eliminated by distributing its probability mass among all other categories, corresponding to their individual weights.

turn correlations. The parameter $\rho_{X_r,R}$, which determines the correlation between the firms' asset returns and the term structure of interest rates, is set equal to $\rho_{X_r,R} = -0.05$. This value lies within the range of correlation parameters estimated in recent empirical studies of structural credit risk models.[12] The credit spreads are set equal to the credit spread means determined by Kiesel et al. (2003). Afterwards, it is assumed that the portfolio consists of $N = 500$ European call options with counterparty risk on (default) risk-free zero coupon bonds, which are written by 500 different counterparties, but are otherwise identical. The parameters of the risk-free interest rates, the recovery rate, the transition matrix, the asset return correlation parameters as well as the correlation parameter between the asset returns and the risk-free interest rates are chosen as above. The simulations are done for the homogeneous initial ratings $\eta_0 \in \{\text{Aa, Baa, B}\}$. The expiration date of the options is set equal to $T^C = 2$, and the exercise price E is chosen as the $(t = 2)$-forward price of the underlying risk-free zero coupon bond. The risk premia needed for computing the one-year risk-neutral default probabilities are taken from Jarrow et al. (1997, p. 518). All calculations are done on a standard Pentium 4 computer with 1.70 GHz and 512 MB RAM using Visual Basic.

5.2 Results for a Portfolio of Defaultable Zero Coupon Bonds

In the following, the percentiles $\alpha_{p\%}(L(H))$ of the credit portfolio loss variable $L(H)$ of a portfolio of defaultable zero coupon bonds are computed for $p \in \{95\%, 99\%, 99.9\%, 99.98\%\}$. In each case, this is done with and without an application of the IS techniques. Repeating these computations several times allows to calculate the standard error of the percentile estimators. Based on these standard errors, the ratio of the standard error of the percentile estimator without an application of IS and the respective standard error of the percentile estimator with an application of IS is computed. These ratios allow to evaluate the effectiveness of the IS technique.

As an initial guess for the percentiles y, which we are looking for, the percentiles resulting from a crude pre-Monte Carlo simulation with a very low number of simulation runs (e.g., 10,000) could be used. These initial guesses are needed for computing the IS distributions of the systematic risk factors. The optimal (conditional) parameters $\theta_y(\mathrm{Z}, \mathrm{X})$ depend on them, too. The exact percentiles are calculated by a simple bisection method. For this, in each itera-

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12 See Eom et al. (2004, table 1, p. 505) and Lyden and Saraniti (2000, table 6, p. 38).

tive step, the arguments in the indicator function of the IS estimators (18) and (43), respectively, are modified until the excess probability estimator equals one minus the confidence level with the desired precision. However, the value of y, on which the parameter $\theta_y(\mathrm{Z},\mathrm{X})$ and the IS means of the systematic risk factors depend, is not altered during the iteration for finding a percentile. In the case that the third step, as described in section 4.3, is also employed, an additional pre-simulation is carried out. The resulting guesses y^* for the percentiles are used for calculating the IS means and IS variances of the market risk factors (see (35), (36), and (39)). For this second pre-simulation, the future ratings of the obligors are set equal to their current ratings; consequently, portfolio losses are only due to changes in the market risk factors.

Table 1 shows the standard error ratios when the two step-estimator (18) is employed. The most important observation is that even for a credit portfolio model with integrated market risk the two step-IS technique is indeed capable of reducing the standard error of the percentile estimators substantially. Only for the very good credit quality Aa and the extreme asset return correlation $\rho_R = 40\%$ the reduction of the standard error is small. There is also evidence for a small sample bias for extremely large asset correlations and/or very high confidence levels. As expected, the reduction of the standard error caused by IS is generally larger, the higher the confidence level of the percentile estimator is. However, no clear dependence of the standard error reduction on the credit quality or the asset return correlation can be observed.[13]

– insert table 1 about here –

Under mild smoothness requirements on the distribution function of the credit portfolio loss $L(H)$ in the neighborhood of the $\alpha_{p\%}(L(H))$, the sample $p\%$-percentile is asymptotically normally distributed with mean $\hat{\alpha}_{p\%}(L(H))$ and variance $p\cdot(1-p)/D\cdot(f(\alpha_{p\%}(L(H))))^2$

[13] To exclude that this missing dependence of the standard error reduction on the credit quality or the asset return correlation is due to differently exact percentile estimates resulting from the pre-simulations, the exact percentile values are employed for computing the IS means, the optimal (conditional) thetas, and as initial guesses for the iterations. The sensitivity of the standard error reduction effect on these initial guesses was also tested: the same computations, as shown in table 1, were repeated using the exact percentiles times a factor 0.75 as initial guesses. Of course, the IS means for the systematic risk factors changed, but nevertheless, in general, the reduction effect was still substantial (without table). These observations are in line with the results of GL (2005) for the default mode model. However, using too large percentile estimates as initial guesses (e.g., the exact percentiles times a factor 1.25) can cause an instability and a missing convergence of the two step-IS approach. GL (2005, p. 1650) do not mention this problem. They only note that one can use the same samples of the loss variable $L(H)$ to estimate excess probabilities $P(L(H) > x)$ at values of x larger than y, but do not comment on the reverse case $x < y$.

(see Serfling (1980, pp. 77)). $f(\cdot)$ denotes the probability density function of the credit portfolio loss $L(H)$, which is assumed to be strictly positive at a neighborhood of the sample percentile $\hat{\alpha}_{p\%}(L(H))$. The nominator $p\cdot(1-p)$ is the variance of the indicator function $1_{\{L(H)>\alpha_{p\%}(L(H))\}}$. Based on the asymptotic normality of $\hat{\alpha}_{p\%}(L(H))$, a large sample $(1-\beta)$-confidence interval for the $p\%$-percentile can be given (see Glasserman (2004, p. 490)):

$$\hat{\alpha}_{p\%}(L(H)) \pm z_{\beta/2}\frac{\sqrt{p\cdot(1-p)}}{\sqrt{D}\cdot f(\alpha_{p\%}(L(H)))} \tag{48}$$

where $z_{\beta/2}=\Phi^{-1}\left(1-\beta/2\right)$. If the portfolio density function $f(\cdot)$ is continuous at $\hat{\alpha}_{p\%}(L(H))$, the confidence interval given by the boundaries in (48) remains asymptotically valid with $f(\alpha_{p\%}(L(H)))$ replaced by $f(\hat{\alpha}_{p\%}(L(H)))$. However, for applying the above confidence interval, the density function at $\hat{\alpha}_{p\%}(L(H))$ has to be estimated. To avoid this, the sample standard error of $\hat{\alpha}_{p\%}(L(H))$, which results from several repetitions of the Monte Carlo simulation, is used in (48). These confidence intervals with $\beta=0.01$ can be seen in table 1. (48) also shows how to interpret the standard error ratios of the percentile estimators. If, for example, the standard error of a percentile estimator without IS is 10 times larger than the standard error with IS, then without IS, 10^2 times more simulation runs D are needed to achieve the same precision, measured by the standard deviation of the percentile estimator, as with IS. Table 1 also shows that the Monte Carlo simulation combined with the IS technique takes more time than the naïve Monte Carlo simulation. The reason is that, for each simulation run, the optimization problem (14) has to be solved for finding the optimal parameter θ.[14] In contrast, the optimization problem (17) for finding the IS means for the systematic risk factors Z and X_r has only to be solved once. Another reason why the IS approach takes more time is that, due to the different IS means of the systematic risk factors for different confidence levels, the simulations for the estimation of the different percentiles have to be done separately for each confidence level.[15]

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14 The increase in run time is larger, the larger the extent of inhomogeneity in the portfolio is. This disadvantage could be avoided if only a one step-IS procedure is used (see the literature review in section 2). However, in the end, the trade-off between the variance reduction effect and the run time increase has to be compared for one step- and two step-IS algorithms.

15 Alternatively, to save computation time, one set of IS means could be used for the estimation of all percentiles corresponding to 'higher' confidence levels. GL (2005, p. 1650) report that the variance reduction effect is relatively insensitive with respect to the choice of the initial percentile guess on which the IS means (and the conditional optimal theta values) depend.

Table 2 shows the relative importance of IS for the systematic credit risk factor Z and the interest rate factor X_r: for high quality portfolios with a low stochastic dependence between the credit quality changes of the obligors, IS for the interest rate factor X_r is more important, whereas for low credit qualities and/or high asset return correlations, IS for the systematic credit risk factor Z is essential.

– insert table 2 about here –

It is also tested how effective it is to use only the IS transition probabilities (10) while setting the IS means of the systematic risk factors equal to zero. Doing this, a reduction of the standard errors of the percentile estimators can only be observed when the factor weights ρ_R and $\rho_{X_r,R}$ and the interest rate volatility σ_r are small (without table). This finding is in line with the theoretical results of GL (2005) who show, on the basis of a one-factor model applied to a homogeneous portfolio, that for larger asset return correlations, IS for the systematic risk factors is more important than the IS for the conditional transition probabilities. In contrast to GL's standard default mode model, which they employ for their numerical experiments, in the integrated market and credit portfolio model, which is used here, correlations of the instrument values are also caused by a common dependence on the risk-free discount factor whenever the interest rate volatility σ_r is positive.

Next, the effectiveness of the two step-IS technique for estimating the expected shortfall, which is – in contrast to the VaR – a coherent risk measure, is tested. It can be shown that the same IS means for the systematic risk factors and the same θ values that are optimal for estimating the VaR are also optimal (in the sense of section 4.1) for estimating the expected shortfall. Table 3 shows the standard error ratios for $E^P\left[L(H)\middle|L(H)>y\right]$ with $y=\hat{\alpha}_{p\%}(L(H))$ ($p\in\{95\%,99\%,99.9\%,99.98\%\}$). In general, the standard error reduction effect is substantially strengthened for this risk measure. Without an application of IS, even the mean of the expected shortfall estimator corresponding to the largest confidence level $p=99.98\%$ over all 350 repetitions is so poor that it is not of any use (without table). However, this is not surprising because with 10,000 simulation runs, on average, only two realizations of the credit portfolio value are larger than the threshold $y=\hat{\alpha}_{99.98\%}(L(H))$. Thus, on average, only two out of 10,000 simulation runs are relevant for the computation of the expected shortfall when IS is not used.

– insert table 3 about here –

Next, the influence of the model parameterization and the homogeneity assumption on the standard error reduction effect is tested. As table 4 shows, the standard error ratios are rather robust with respect to changes in the amount of interest rate risk, the correlation between the asset returns and the risk-free interest rates, the number of obligors, or the degree of homogeneity in the portfolio. Some standard error ratios are larger than in the base case setting, others are lower, but no systematic difference can be observed (compare with table 1). Even for the very good credit quality Aa and the extreme asset return correlation $\rho_R = 40\%$, a substantial variance reduction effect, which is larger than that one in the base case setting, can be observed for some parameterizations (without table). This indicates that, to some extent, the effectiveness of the two step-IS technique depends on the parameters of the model.

– insert table 4 about here –

The effectiveness of the two step-IS technique is also tested when the first step, the IS for the conditional transition probabilities, is only done while determining the IS means for the systematic risk factors, but not while the actual simulations later. Surprisingly, in many cases, a similar or only slightly smaller reduction of the standard error of the percentile estimators than with the full two step-IS technique can be observed (without table). Checking the optimal theta values, which result from setting the systematic risk factors equal to their IS means, this result is not too surprising any more. Frequently, these optimal theta values are zero or slightly above zero. This explains why the effect from setting theta equal to zero during the simulations is not too large. However, this observation raises doubt whether a two step-IS technique is really necessary or whether simpler one step-IS techniques[16] might not be as effective as the two step-IS approach.

Finally, the three step-IS estimator is implemented. The IS mean of the systematic credit risk factor Z is calculated according to (47), whereas the computation of the IS mean and IS variance of the interest rate factor X_r is based on (35), (36), and (39). As the second derivative $\partial^2 L^{wtr}(X_r, H)/(\partial X_r)^2\big|_{X_r = E^P[X_r]=0}$ is very small, the parameter λ_1 is nearly zero. This implies that the IS variance (36) remains, compared to the original probability measure, nearly unchanged one. Table 5 shows that for almost all considered credit qualities, asset return corre-

16 See the literature review in section 2.

lations, and confidence levels, the three step-IS technique yields worse standard error reductions than the two step-IS approach. Only for an initial rating Aa and an asset return correlation of 10%, which is the scenario in which interest rate risk has its largest importance (see table 2), the results are comparable with the two step-IS technique. One reason for this worse performance of the three step-IS technique is that the approach for computing the optimal parameter θ when calculating the IS mean of Z is not identical with the approach for computing this parameter during each of the simulation runs. In the former case, the interest rate factor is assumed to be non-stochastic and set equal to its IS mean. In contrast, in the latter case, θ is chosen as a function of the realizations of the systematic risk factors Z and X_r, which can both vary in a free manner according to their probability distribution. Hence, when determining the optimal IS mean of Z only a sub-optimal parameter θ (and not the optimal value characterized by (14)) is employed. From this the sub-optimality of the computed IS mean of Z follows. Furthermore, the IS mean of the interest rate factor X_r chosen according to (35) hardly depends on the initial rating of the obligors. The consequence is that no reduction of the IS mean of X_r takes place when more importance has to be put on the credit risk factor Z. For example, for the initial rating B, an asset return correlation of $\rho_R = 10\%$, and a confidence level of 99.98%, the optimal two step-IS means would be $(\mu_Z^{2\,step}, \mu_{X_r}^{2\,step}) = (-3.5, 1.1)$. However, the IS means employed for the three step-IS technique are $(\mu_Z^{3\,step}, \mu_{X_r}^{3\,step})$ $= (-2.6, 3.5)$. The consequence is that a reduction of the standard error of the percentile estimators can hardly be observed any more for this scenario. In contrast, for the initial rating Aa, $\rho_R = 10\%$, and a confidence level of 99.98%, the difference between the optimal two step-IS means $(\mu_Z^{2\,step}, \mu_{X_r}^{2\,step}) = (0, 3.6)$ and those IS means $(\mu_Z^{3\,step}, \mu_{X_r}^{3\,step}) = (-0.6, 3.5)$ employed for the three step-IS technique is not too large. As a consequence, for this scenario, the standard error reduction effect produced by the three step-IS technique is still substantial as table 5 shows.

– insert table 5 about here –

5.3 Results for a Portfolio of European Call Options with Counterparty Risk

Next, the two step-IS technique is applied to an interest rate option portfolio. However, as the main reason for losses in an option portfolio is a negative development of the market risk factors underlying the options, the risk measures are hardly sensitive to changes in the credit quality of the issuer or the asset return correlation. These findings are in line with those of

Duffie and Pan (2001, table 5, figure 6a). For the two step-IS technique, the consequence is that the optimization problem (14) might not be well defined. This optimization problem has a solution $\theta \in (0,\infty)$ only for initial guesses y of the percentile, which are larger than the conditional expected loss and smaller than the conditional maximum loss (see (26)). However, as table 6 shows, this interval can be rather small for an option portfolio, especially for higher confidence levels. And even if the initial guess is in the interval (26), the optimal parameter θ fulfilling (14) can be very large, which can cause overflow problems during the simulations. For example, setting the systematic risk factors (Z, X_r) equal to $(0, 3.5)$ for an initial rating Aa and a confidence level of 99.98% yields non-zero values for θ only for $y > 1.723804$. However, for the slightly increased initial guess $y = 1.723805$, the optimal value for θ is already $14{,}580$. The reason for these large θ values is the relatively small influence of rating transitions on the loss of the option when the interest rate factor X_r is around its IS mean.

– insert table 6 about here –

Despite these difficulties, the performance of the two step-IS technique is also tested for an interest rate option portfolio. For this, a maximum value for the parameter θ is defined: whenever the true value is larger than 100, $\theta = 100$ is set. As table 7 shows, the two step-IS technique still yields a substantial reduction of the standard error of the percentile estimators.[17]

– insert table 7 about here –

As the loss percentiles of the interest rate option portfolio are hardly sensitive to the initial rating of the obligors or the asset return correlation, one might think about applying a one step-IS approach consisting only of the IS technique developed for pure market risk portfolios (see section 4.3). Considering the homogeneity of the option portfolio and the fact that there is only one market risk factor in this numerical example, the optimization problem (39) for finding the optimal parameter $\theta_{y^*}^{wtr}$ is well defined for initial guesses y^* of the loss percentiles of a default risk-free portfolio of interest rate options out of the interval (see (42)):

[17] However, for the option portfolio, more frequently the missing convergence of the iteration, mentioned already before, is observed, even if the exact percentiles are used as initial guesses. In these cases, the optimal parameter $\theta_y(\mathrm{Z}^{(d)}, \mathrm{X}^{(d)})$ is too often too large which implies that the conditional likelihood ratio $\exp(-\theta_y(\mathrm{Z}^{(d)}, \mathrm{X}^{(d)}) L(H)^{(d)} + \psi_{L(H)|\mathrm{Z}^{(d)}, \mathrm{X}^{(d)}}(\theta_y(\mathrm{Z}^{(d)}, \mathrm{X}^{(d)})))$ is too often almost or identical to zero so that not enough probability mass is produced. This problem occurs when the maximum value for theta is chosen too large (e.g., $\theta^{\max} = 15{,}000$). In the numerical experiments, it can also be observed that the standard error reduction effect occasionally depends on the chosen maximum value for theta (without table).

$$y^* \in \left(\lambda_1, -b_1^2/4\lambda_1\right). \tag{49}$$

For the parameter values of the numerical example, this interval is $(-0.2672, 1.5313)$. However, only the initial guesses of the loss percentiles corresponding to a confidence level of 95% are smaller than the upper boundary 1.5313. The consequence is that the optimization problem (39) is not well defined for all larger confidence levels: the optimal parameter $\theta_{y^*}^{wtr}$ converges to infinity so that the mean of the market risk factor X_r under the IS distribution is $-b_1/2\lambda_1$ and the standard deviation goes to zero (see (35), (36)). Thus, the IS distribution of the market risk factor is a Dirac distribution. The reason for the failure of this IS technique when applied to an option portfolio is the bad approximation quality of a delta-gamma approximation over a one-year risk horizon.[18] However, as explained in section 4.3, the whole IS technique is based on this approximation. The bad approximation quality for those values of X_r which are necessary to produce loss percentiles corresponding to large confidence levels can also be seen from figure 1.

– insert figure 1 about here –

The only way to apply the IS technique of section 4.3 to the interest rate option portfolio is to employ that IS distribution of X_r which is optimal for the 95%-loss percentile also for the estimation of larger loss percentiles. However, as table 8 shows, the resulting reduction of the standard errors of the percentile estimators is poor. Only for the 95%-loss percentile, the reduction effect is similar to that one which results from an application of the two step-IS technique (see table 7).

– insert table 8 about here –

6 Conclusions

In this paper, the benefit which results from applying importance sampling techniques to an integrated market and credit portfolio model is analyzed. It is shown in detail how the two step-IS technique of GL (2005) can be adjusted to the general integrated market and credit portfolio model of section 3.1. As GL (2005) employ IS for the probability distribution of the systematic risk factors as well as for the conditional default probabilities, this approach was expected to be especially effective. Furthermore, it is discussed how an IS approach originally

[18] In contrast, Glasserman et al. (2000) estimate in their numerical examples 99% – loss percentiles over a risk horizon of only ten days.

developed for pure market risk portfolio models can be combined with the two step-IS approach to build up a potentially even more effective three step-IS technique. The effectiveness of the various approaches when estimating large percentiles of the credit portfolio loss variable is tested by means of numerical experiments.

The main result is that the two step-IS technique of GL (2005) originally developed for pure default mode credit portfolio models can basically also be applied to integrated market and credit portfolio models. As expected, this technique can substantially reduce the dispersion of the percentile estimators, even in the context of an integrated market and credit portfolio model.

The combination of the two step-IS technique of GL (2005) with an IS technique originally developed for pure market risk portfolio models is less effective than the simpler two step-IS method when applied to a bond portfolio (at least for the chosen implementation of the method in this paper). This result is rather unexpected. For an interest rate option portfolio, the IS technique originally developed for pure market risk portfolio models was not applicable at all due to the bad quality of the delta-gamma approximation over a long risk horizon of one year.

However, the previous analysis also reveals that the two step-IS technique of GL (2005) when adjusted to an integrated market and credit portfolio model is not necessarily the ideal choice. Due to the market risk dependency of the losses, numerical problems can arise during the solution of the optimization problem (14). This problem could be observed for an option portfolio with counterparty risk and made an ad-hoc adjustment of the IS technique necessary. Furthermore, the solution of the optimization problem (14) in each simulation run increases the computational burden of this IS technique compared to simpler one step-IS methods (see the overview in section 2), in particular in the case of very inhomogeneous credit portfolios. Thus, in future research, the performance of these one step-IS approaches with respect to their ability to reduce the standard error of the percentile estimators as well as the required computational time should be compared with the performance of the two step-IS technique employed in this paper. Doing this performance comparison, the effect of the number of systematic risk factors on each method's performance should also be analyzed.

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TABLES:

Table 1:
Standard Error Ratios for Percentile Estimators with the Two Step-IS Technique

	$\alpha_{99.98\%}(L)$	$\alpha_{99.9\%}(L)$	$\alpha_{99\%}(L)$	$\alpha_{95\%}(L)$	$\alpha_{99.98\%}(L)$	$\alpha_{99.9\%}(L)$	$\alpha_{99\%}(L)$	$\alpha_{95\%}(L)$
	Aa, $\rho_R = 10\%$				Aa, $\rho_R = 40\%$			
MC^{IS}	17.59	15.41	11.68	8.34	17.92	15.84	11.78	8.37
	*0.0249	*0.0265	*0.0294	*0.0335	*0.7034	*1.0172	*0.0659	*0.0370
	[17.53;17.65]	[15.34;15.48]	[11.60;11.76]	[8.25;8.43]	[16.11;19.73]	[13.22;18.46]	[11.61;11.95]	[8.27;8.47]
rt	168				157			
MC	17.90	15.31	11.67	8.33	20.17	15.75	11.79	8.38
	*1.0333	*0.4478	*0.1812	*0.1047	*2.9906	*0.5056	*0.1867	*0.1072
	[15.24;20.56]	[14.16;16.46]	[11.20;12.14]	[8.06;8.60]	[12.47;27.87]	[14.45;17.05]	[11.31;12.27]	[8.10;8.66]
rt	53				52			
ratio	41.5	16.9	6.2	3.1	4.3	0.5	2.8	2.9
	Baa, $\rho_R = 10\%$				Baa, $\rho_R = 40\%$			
MC^{IS}	19.67	17.04	12.87	9.29	50.25	32.51	15.57	9.84
	*0.0599	*0.0439	*0.0334	*0.0361	*0.2415	*0.1796	*0.2794	*0.0757
	[19.52;19.82]	[16.93;17.15]	[12.78;12.96]	[9.20;9.38]	[49.63;50.87]	[32.05;32.97]	[14.85;16.29]	[9.65;10.03]
rt	128				110			
MC	20.03	16.95	12.85	9.27	53.96	31.98	15.54	9.83
	*1.2745	*0.5036	*0.1983	*0.1128	*10.7274	*2.9405	*0.4141	*0.1439
	[16.75;23.31]	[15.65;18.25]	[12.34;13.36]	[8.98;9.56]	[26.33;81.59]	[24.41;39.55]	[14.47;16.61]	[9.46;10.20]
rt	53				54			
ratio	21.3	11.5	6.0	3.1	44.4	16.4	1.5	1.9
	B, $\rho_R = 10\%$				B, $\rho_R = 40\%$			
MC^{IS}	70.33	60.27	44.64	32.66	151.10	133.51	93.51	55.97
	*0.1115	*0.1084	*0.1065	*0.1030	*0.1584	*0.2416	*0.3318	*0.3315
	[70.04;70.62]	[59.99;60.55]	[44.37;44.91]	[32.39;32.93]	[150.69;151.51]	[132.89;134.13]	[92.66;94.36]	[55.12;56.82]
rt	87				73			
MC	71.99	60.00	44.61	32.67	152.30	132.52	93.37	55.92
	*4.9163	*1.8952	*0.6798	*0.3345	*6.5357	*4.0189	*2.0336	*1.0868
	[59.33;84.65]	[55.12;64.88]	[42.86;46.36]	[31.81;33.53]	[135.47;169.13]	[122.17;142.87]	[88.13;98.61]	[53.12;58.72]
rt	54				53			
ratio	44.1	17.5	6.9	3.3	41.3	16.6	6.1	3.3

Notes: MC^{IS}: Monte Carlo simulation of the loss variable $L(H)$ for a portfolio of defaultable zero coupon bonds with the two step-IS technique; MC: Monte Carlo simulation without IS; *: standard error of the percentile estimators; [ ]: 99%-confidence interval of the percentile estimators; rt: run time in seconds (for MC^{IS} without the one-time solution of the optimization problem (17) for finding the IS means); ratio: standard error ratios, defined as the standard error of the percentile estimator without an application of IS divided by the standard error of the respective estimator with IS. The computations of the standard errors are based on 350 repetitions of the simulations (with and without IS), whereby each simulation consists of 10,000 simulation runs. All exhibited percentiles are mean values of these 350 repetitions. *Notation*: ρ_R: asset return correlation. *Parameters*: $N = 500$, $F = 1$, $T = 3$, $H = 1$, $\rho_{X_r,R} = -0.05$, $\delta = 0.538$, $\kappa = 0.4$, $\theta = 0.06$, $\sigma_r = 0.01$, $\lambda = 0.5$, $r(0) = 0.06$.

Table 2:
Relative Importance of the IS technique for the Two Systematic Risk Factors

confidence level	Aa		Baa		B	
	$\rho_R = 10\%$	$\rho_R = 40\%$	$\rho_R = 10\%$	$\rho_R = 40\%$	$\rho_R = 10\%$	$\rho_R = 40\%$
			no IS for Z			
99.98%	39.5	1.5	4.3	0.9	0.9	1.6
99.9%	16.9	0.4	3.3	1.1	1.2	0.9
99%	6.4	1.2	3.5	1.1	1.1	1.0
95%	3.0	2.6	2.6	1.1	1.2	1.0
			no IS for X_r			
99.98%	1.0	4.5	1.5	34.8	21.0	30.6
99.9%	0.9	0.9	0.8	12.4	9.2	14.0
99%	1.0	1.0	0.9	1.0	3.8	5.6
95%	1.0	1.0	1.1	0.6	2.4	3.0

Notes: Standard error ratios, defined as the standard error of the percentile estimator of the loss variable $L(H)$ for a portfolio of defaultable zero coupon bonds without an application of IS divided by the standard error of the respective estimator with the two step-IS technique, are shown. In the first case, the IS mean of the systematic credit risk factor Z is set equal to zero and only the optimal IS mean for the interest rate factor X_r is employed. In the second case, the IS mean of X_r is set equal to zero. The computations of the standard errors are based on 350 repetitions of the simulations (with and without IS), whereby each simulation consists of 10,000 simulation runs. *Notation*: ρ_R: asset return correlation. *Parameters*: $N = 500$, $F = 1$, $T = 3$, $H = 1$, $\rho_{X_r,R} = -0.05$, $\delta = 0.538$, $\kappa = 0.4$, $\theta = 0.06$, $\sigma_r = 0.01$, $\lambda = 0.5$, $r(0) = 0.06$.

Table 3:
Standard Error Ratios for Expected Shortfall Estimators with the Two Step-IS Technique

confidence level	Aa $\rho_R = 10\%$	Aa $\rho_R = 40\%$	Baa $\rho_R = 10\%$	Baa $\rho_R = 40\%$	B $\rho_R = 10\%$	B $\rho_R = 40\%$
99.98%	217.2	1.1	62.9	100.3	207.5	276.7
99.9%	53.6	0.3	18.5	38.2	44.2	49.6
99%	11.4	0.8	9.3	4.6	11.2	10.5
95%	5.5	2.1	5.1	4.1	5.3	5.1

Notes: Standard error ratios, defined as the standard error of the estimator for the expected shortfall $E^P\left[L(H)\middle| L(H) > \alpha_p\left(L(H)\right)\right]$ for a portfolio of defaultable zero coupon bonds without an application of IS divided by the standard error of the respective estimator with the two step-IS technique, are shown. The computations of the standard errors are based on 350 repetitions of the simulations (with and without IS), whereby each simulation consists of 10,000 simulation runs. *Notation*: ρ_R: asset return correlation. *Parameters*: $N = 500$, $F = 1$, $T = 3$, $H = 1$, $\rho_{X_r,R} = -0.05$, $\delta = 0.538$, $\kappa = 0.4$, $\theta = 0.06$, $\sigma_r = 0.01$, $\lambda = 0.5$, $r(0) = 0.06$.

Table 4:
Robustness Checks for the Standard Error Ratios for Percentile Estimators with the Two Step-IS Technique

confidence level	Aa			Baa			B		
interest rate volatility	$\sigma_r = 0$	$\sigma_r = 0.005$	$\sigma_r = 0.02$	$\sigma_r = 0$	$\sigma_r = 0.005$	$\sigma_r = 0.02$	$\sigma_r = 0$	$\sigma_r = 0.005$	$\sigma_r = 0.02$
99.98%	47.2	37.3	40.0	48.9	41.6	35.1	40.3	40.3	42.3
99.9%	16.4	15.9	16.9	17.6	14.3	14.9	17.3	17.2	16.5
99%	7.2	5.6	6.1	6.4	5.6	5.8	6.5	6.6	5.9
95%	2.7	3.1	3.0	3.4	3.1	3.1	3.4	3.3	2.9

correlation between asset returns and interest rates	$\rho_{X_r,R} = -0.25$	$\rho_{X_r,R} = 0.25$	$\rho_{X_r,R} = -0.25$	$\rho_{X_r,R} = 0.25$	$\rho_{X_r,R} = -0.25$	$\rho_{X_r,R} = 0.25$
99.98%	41.1	40.9	40.9	35.4	43.2	42.9
99.9%	16.5	15.2	17.6	16.6	16.7	17.3
99%	6.0	6.6	5.8	6.1	5.7	6.7
95%	3.2	3.3	3.0	3.4	3.0	3.2

number of obligors	$N = 50$	$N = 50$	$N = 50$
99.98%	40.2	35.2	38.4
99.9%	18.3	13.4	17.2
99%	6.5	6.2	7.2
95%	2.9	3.2	3.3

inhomogeneous portfolio composition	
99.98%	39.8
99.9%	15.9
99%	5.6
95%	2.9

Notes: Standard error ratios, defined as the standard error of the percentile estimator of the loss variable $L(H)$ for a portfolio of defaultable zero coupon bonds without an application of IS divided by the standard error of the respective estimator with the two step-IS technique, are shown. The computations of the standard errors are based on 350 repetitions of the simulations (with and without IS), whereby each simulation consists of 10,000 simulation runs. The inhomogeneous portfolio contains the following numbers n_j of bonds, whose issuers have the initial rating j and whose face values are F_j: $(n_{Aaa}, F_{Aaa}) = (15,1)$, $(n_{Aa}, F_{Aa}) = (25,1)$, $(n_A, F_A) = (65,5)$, $(n_{Baa}, F_{Baa}) = (155,10)$, $(n_{Ba}, F_{Ba}) = (165,10)$, $(n_B, F_B) = (55,5)$, $(n_{Caa}, F_{Caa}) = (20,1)$. *Parameters (as far as not otherwise indicated)*: $N = 500$, $F = 1$, $T = 3$, $H = 1$, $\rho_R = 0.1$, $\rho_{X_r,R} = -0.05$, $\delta = 0.538$, $\kappa = 0.4$, $\theta = 0.06$, $\sigma_r = 0.01$, $\lambda = 0.5$, $r(0) = 0.06$.

Table 5:
Standard Error Ratios for Percentile Estimators with the Three Step-IS Technique

confidence level	Aa		Baa		B	
	$\rho_R = 10\%$	$\rho_R = 40\%$	$\rho_R = 10\%$	$\rho_R = 40\%$	$\rho_R = 10\%$	$\rho_R = 40\%$
99.98%	36.6	3.2	13.1	1.5	2.6	1.8
99.9%	17.0	2.3	9.0	0.9	1.3	1.0
99%	6.2	3.6	5.7	0.8	1.4	0.7
95%	3.2	2.4	3.1	2.4	1.6	1.0

Notes: Standard error ratios, defined as the standard error of the percentile estimator of the loss variable $L(H)$ for a portfolio of defaultable zero coupon bonds without an application of IS divided by the standard error of the respective estimator with the three step-IS technique, are shown. The computations of the standard errors are based on 350 repetitions of the simulations (with and without IS), whereby each simulation consists of 10,000 simulation runs. The IS means of the systematic credit risk factor Z are calculated according to (47), whereas the computations of the IS means and variances of the interest rate factor X_r are based on (35), (36), and (39). *Notation*: ρ_R : asset return correlation. *Parameters*: $N = 500$, $F = 1$, $T = 3$, $H = 1$, $\rho_{X_r,R} = -0.05$, $\delta = 0.538$, $\kappa = 0.4$, $\theta = 0.06$, $\sigma_r = 0.01$, $\lambda = 0.5$, $r(0) = 0.06$.

Table 6:
Admissible Intervals for the Initial Guess y and Optimal θ Values

	confidence level 99.98%			confidence level 95%	
	exact percentile: 1.725242			exact percentile: 1.378290	
X_r	interval	θ^{opt}	X_r	interval	θ^{opt}
3.1	(1.700319; 1.721899)	∞	1.0	(1.004624; 1.347275)	∞
3.2	(1.707588; 1.725813)	> 15,000	1.1	(1.075470; 1.385424)	> 15,000
3.3	(1.713848; 1.729184)	> 15,000	1.2	(1.141300; 1.420872)	> 15,000
3.4	(1.719217; 1.732075)	> 15,000	1.5	(1.310348; 1.511902)	> 15,000
3.5	(1.723804; 1.734545)	> 15,000	1.6	(1.357837; 1.537475)	> 15,000
3.6	(1.727708; 1.736648)	0	*1.7*	(1.401236; 1.560845)	0
3.7	(1.731017; 1.738430)	0	1.8	(1.440758; 1.582127)	0

Notes: Table 6 shows for a homogeneous portfolio of European call options with counterparty risk on (default) risk-free zero coupon bonds, whereby the issuers of the options have an initial rating of Aa and an asset return correlation of $\rho_R = 10\%$, the admissible intervals (according to (26)) for the initial percentile guess y. Furthermore, the solution θ of the optimization problem (14) when y is set equal to the exact percentile is exhibited for various scenarios of the systematic risk factors. The systematic credit risk factor Z is set equal to zero, which is its IS mean in all cases. The values of the interest rate factor X_r are chosen around its IS mean (cursive in the above table). For computing the IS means as well as for the above calculations, the maximum value of θ is set equal to 15,000. *Parameters*: $N = 500$, $F = 1$, $T = 3$, $T^C = 2$, $H = 1$, $E = 0.934498$, $\rho_{X_r,R} = -0.05$, $\delta = 0.538$, $\kappa = 0.4$, $\theta = 0.06$, $\sigma_r = 0.01$, $\lambda = 0.5$, $r(0) = 0.06$.

Table 7:
Standard Error Ratios for Percentile Estimators of an Interest Rate Option Portfolio with the Two Step-IS Technique

	$\alpha_{99.98\%}(L)$	$\alpha_{99.9\%}(L)$	$\alpha_{99\%}(L)$	$\alpha_{95\%}(L)$	$\alpha_{99.98\%}(L)$	$\alpha_{99.9\%}(L)$	$\alpha_{99\%}(L)$	$\alpha_{95\%}(L)$
	Aa, $\rho_R = 10\%$				Aa, $\rho_R = 40\%$			
MC^{IS}	1.7255	1.6996	1.5938	1.3780	1.7254	1.6995	1.5936	1.3776
	*0.0002	*0.0004	*0.0012	*0.0028	*0.0002	*0.0004	*0.0013	*0.0030
MC	1.7266	1.6979	1.5929	1.3775	1.7268	1.6981	1.5935	1.3775
	*0.0072	*0.0069	*0.0082	*0.0089	*0.0073	*0.0071	*0.0079	*0.0094
ratio	36.3	17.2	6.7	3.2	33.5	16.0	5.9	3.2
	Baa, $\rho_R = 10\%$				Baa, $\rho_R = 40\%$			
MC^{IS}	1.7206	1.6948	1.5894	1.3744	1.7206	1.6948	1.5894	1.3743
	*0.0002	*0.0004	*0.0013	*0.0030	*0.0002	*0.0004	*0.0013	*0.0029
MC	1.7216	1.6927	1.5885	1.3738	1.7219	1.6930	1.5887	1.3738
	*0.0073	*0.0073	*0.0081	*0.0092	*0.0073	*0.0076	*0.0081	*0.0090
ratio	35.1	17.3	6.1	3.1	35.4	17.5	6.2	3.1
	B, $\rho_R = 10\%$				B, $\rho_R = 40\%$			
MC^{IS}	1.7046	1.6801	1.5796	1.3740	1.7047	1.6803	1.5802	1.3751
	*0.0002	*0.0004	*0.0013	*0.0029	*0.0002	*0.0004	*0.0013	*0.0028
MC	1.7057	1.6787	1.5792	1.3738	1.7058	1.6785	1.5794	1.3751
	*0.0067	*0.0070	*0.0074	*0.0086	*0.0068	*0.0072	*0.0076	*0.0087
ratio	32.6	16.0	5.7	2.9	35.1	17.9	6.1	3.2

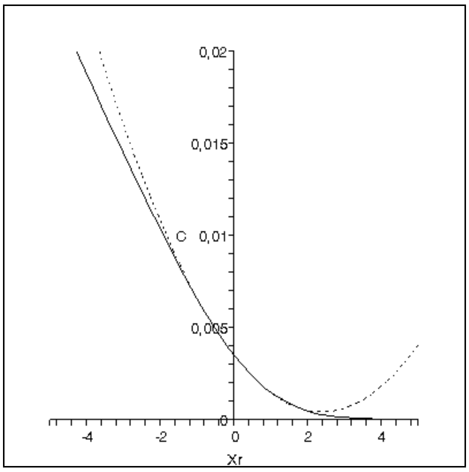
Notes: MC^{IS}: Monte Carlo simulation of the loss variable $L(H)$ for a portfolio of European call options with counterparty risk on (default) risk-free zero coupon bonds with the two step-IS technique; MC: Monte Carlo simulation without IS; *: standard error of the percentile estimators; ratio: standard error ratios, defined as the standard error of the percentile estimator without an application of IS divided by the standard error of the respective estimator with IS. For the computation of the IS means as well as for the simulation, the maximum value for θ is set equal to 100. The computations of the standard errors are based on 350 repetitions of the simulations (with and without IS), whereby each simulation consists of 10,000 simulation runs. All exhibited percentiles are mean values of these 350 repetitions. *Notation*: ρ_R: asset return correlation. *Parameters*: $N = 500$, $F = 1$, $T = 3$, $T^C = 2$, $H = 1$, $E = 0.934498$, $\rho_{X_r,R} = -0.05$, $\delta = 0.538$, $\kappa = 0.4$, $\theta = 0.06$, $\sigma_r = 0.01$, $\lambda = 0.5$, $r(0) = 0.06$.

Table 8:
Standard Error Ratios for Percentile Estimators of an Interest Rate Option Portfolio with the One Step-IS Technique

	Aa		Baa		B	
confidence level	$\rho_R = 10\%$	$\rho_R = 40\%$	$\rho_R = 10\%$	$\rho_R = 40\%$	$\rho_R = 10\%$	$\rho_R = 40\%$
99.98%	1.0	1.1	1.1	1.1	1.1	1.1
99.9%	1.1	1.2	1.2	1.3	1.2	1.2
99%	3.3	3.3	3.0	3.2	3.3	3.1
95%	4.9	4.5	4.7	5.1	4.3	4.8

Notes: Standard error ratios, defined as the standard error of the percentile estimator of the loss variable $L(H)$ for a portfolio of European call options with counterparty risk on (default) risk-free zero coupon bonds without an application of IS divided by the standard error of the respective estimator with an application of the IS technique for the interest rate factor X_r as described in section 4.3, are shown. As the optimization problem (39) is only well defined for the 95%-loss percentile, the IS distribution of X_r, which is optimal for the 95%-loss percentile, is also used for the estimation of the larger loss percentiles. No IS is done for the systematic credit risk factor Z. The computations of the standard errors are based on 350 repetitions of the simulations (with and without IS), whereby each simulation consists of 10,000 simulation runs. *Notation*: ρ_R: asset return correlation. *Parameters*: $N = 500$, $F = 1$, $T = 3$, $T^C = 2$, $H = 1$, $E = 0.93449751$, $\rho_{X_r,R} = -0.05$, $\delta = 0.538$, $\kappa = 0.4$, $\theta = 0.06$, $\sigma_r = 0.01$, $\lambda = 0.5$, $r(0) = 0.06$.

Figure 1:
Accuracy of the Delta-Gamma Approximation for Option Prices



Notes: Figure 1 shows the difference between the exact option price (solid line) and the delta-gamma approximation (dashed line) for the price of a European call option on a (default) risk-free zero coupon bond within the Vasicek model. The exercise price E is chosen as the $(t=2)$ -forward price of the underlying risk-free zero coupon bond. *Parameters*: $F=1$, $T=3$, $T^C=2$, $H=1$, $\kappa=0.4$, $\theta=0.06$, $\sigma_r=0.01$, $\lambda=0.5$.