# TAIL DEPENDENCE FOR ELLIPTICALLY CONTOURED DISTRIBUTIONS

RAFAEL SCHMIDT

Abstract. The relationship between the theory of elliptically contoured distributions and the concept of tail dependence is investigated. We show that bivariate elliptical distributions possess the so-called tail dependence property if the tail of their generating random variable is regularly varying, and we give a necessary condition for tail dependence which is somewhat weaker than regular variation of the latter tail. In addition, we discuss the tail dependence property for some well-known examples of elliptical distributions, such as the multivariate normal, t, logistic, and symmetric generalized hyperbolic distributions.

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Contents



## Introduction

The concept of tail dependence plays an important role among dependence measures used within credit risk modelling. Most of the well-known credit risk models which are used in finance are based on the multivariate normal distribution, due to its tractability in portfolio calculation and simulation. However, many statistical

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papers show that the normal distribution is not capable of exhibiting important properties encountered in credit risk. Some of the insufficiencies of normal distributions are their light tails and the independence of extremal credit default events, where the latter can be modelled by tail-dependence measures. To tackle these problems we focus on the larger class of elliptically contoured distributions (in short: elliptical distributions), which include the multivariate normal and t-distributions as representatives. Many convenient properties of the multivariate normal distribution can be extended to the class of elliptical distributions. Especially for credit risk portfolio modelling we mention the simple way of calculating the Value at Risk of a linear portfolio. We prove that the family of elliptically contoured distributions with regularly varying tails of the generating variate is almost equivalent to the class of elliptically contoured distributions possessing the dependence property for extremal credit default events.

## 1. Tail dependence and regular variation

In this section we introduce the concepts of tail dependence and regularly varying functions. We will state the definitions and some well-known results used in the sequel. Multivariate distributions having the tail dependence property are of special practical interest within credit portfolio modelling since they are able to incorporate dependencies of extremal credit default events. According to Embrechts, McNeil, and Straumann [5] tail dependence plays an important role in extreme value theory, finance, and insurance models. Tail dependence models for multivariate distributions are mostly related to their bivariate marginal distributions. The following approach represents one of many possible definitions of tail dependence.

**Definition 1.1** (Tail dependence, Joe [8], p.33)**.** Let $X = (X_1, X_2)'$ be a 2-dimensional random vector. We say that $X$ is *tail dependent* if

$$\lambda := \lim_{v\to 1^-} \mathbb{P}(X_1 > G_1^{-1}(v) \mid X_2 > G_2^{-1}(v)) > 0; \tag{1.1}$$

where the limit exists and $G_1^{-1},\ G_2^{-1}$ denote the generalized inverse distribution functions of $X_1,\ X_2$. Consequently, we say $X = (X_1, X_2)'$ is *tail independent* if $\lambda$ equals 0. Further, we call $\lambda$ the (upper) tail dependence coefficient.

**Remarks.**

1. Similarly, one may define the lower tail dependence coefficient by
$$\omega := \lim_{v\to 0^+} \mathbb{P}(X_1 \le G_1^{-1}(v) \mid X_2 \le G_2^{-1}(v)).$$
2. Later we will also consider the following dependence measure:
$$\hat{\lambda} := \lim_{x\to G_2^{-1}(1)^-} \mathbb{P}(X_1 > x \mid X_2 > x),$$
which turns out to be useful for absolute asset returns.

The main results of this paper are characterized by regularly varying or O-regularly varying functions, which are defined as follows.

**Definition 1.2.** (Regular and O-regular variation)

1. A measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is called *regularly varying* (at $\infty$) with index $\alpha \in \mathbb{R}$ if for any $t > 0$
$$\lim_{x\to\infty} \frac{f(tx)}{f(x)} = t^\alpha.$$
The class of regularly varying functions with index $\alpha$ is denoted by $RV_\alpha$.
2. A measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is called *O-regularly varying* (at $\infty$) if for any $t \geq 1$
$$0 < \liminf_{x\to\infty} \frac{f(tx)}{f(x)} \leq \limsup_{x\to\infty} \frac{f(tx)}{f(x)} < \infty.$$
The class of O-regularly varying functions is denoted by $OR$.

Thus, regularly varying functions behave asymptotically like power functions. The next well-known uniform convergence theorem (see Resnick [9], p.22) will be used frequently in the following.

**Theorem 1.3.** *Let $f \in RV_\alpha$, $\alpha \in \mathbb{R}$.*

1. *Then*
$$\lim_{x\to\infty} \frac{f(xt)}{f(x)} = t^\alpha$$
*holds locally uniformly on $(0, \infty)$. Moreover, if $\alpha < 0$, then uniform convergence holds on intervals of the form $[b, \infty)$, $b > 0$.*
2. *Let $\varepsilon > 0$. Then there exists $K_\varepsilon \geq 0$ such that*
$$(1-\varepsilon)t^{\alpha-\varepsilon} < \frac{f(xt)}{f(x)} < (1+\varepsilon)t^{\alpha+\varepsilon}$$
*for all $x \geq K_\varepsilon$ and $t \geq 1$.*

Further, we need the following result about the derivative of regularly varying functions.

**Theorem 1.4.** *Suppose that the distribution function $F : \mathbb{R}_+ \to [0, 1]$ is absolutely continuous with density $f$, i.e.,*
$$F(x) = \int_0^x f(t)\, dt, \qquad x \geq 0.$$
*If $\overline{F} = 1 - F \in RV_{-\alpha}$, $\alpha > 0$, and $f$ is monotone, then $f \in RV_{-\alpha-1}$.*

We shall also consider a Wiener-Tauberian Theorem dealing with Mellin transforms and Mellin convolutions which we define now.

**Definition 1.5** (Mellin transform and convolution).

1. Given a measurable kernel $k : (0, \infty) \to I\!R$ we define its *Mellin transform* by

$$\check{k}(z) := \int_0^\infty t^{-z} k(t) \frac{dt}{t} \tag{1.2}$$

for $z \in \mathbf{C}$ such that the integral converges.

2. For measurable functions $f, g : (0, \infty) \to I\!R$ we call $g \overset{M}{*} f$ the *Mellin convolution*, where

$$g \overset{M}{*} f(x) := \int_0^\infty f(x/t) g(t) \frac{dt}{t} \tag{1.3}$$

for those $x > 0$ for which the integral converges.

For proofs and more details regarding regular variation, O-regular variation, and Wiener-Tauberian Theorems we refer the reader to Bingham, Goldie, and Teugels [2], pp. 16, pp. 61, and pp. 193 and Resnick [9], pp. 12.

## 2. Spherical distributions

We begin this section by introducing the family of spherical distributions, which forms a subclass of the family of elliptical distributions. After formulating the main results for the class of spherical distributions we can easily extend them to the more general framework of elliptical distributions. Most of the following notation is along the lines of Fang, Kotz, and Ng [6].

**Definition 2.1.** Let $X$ be an $n$-dimensional random vector. Then $X$ is said to be *spherically distributed* if

$$OX \overset{d}{=} X$$

for every orthogonal matrix $O \in I\!R^{n \times n}$.

One can easily show that $X$ belongs to the class of spherical distributions if and only if its characteristic function $\phi(t)$, $t \in I\!R^n$ has the form

$$\phi(t) = \Phi(t't)$$

for some function $\Phi : I\!R \to I\!R$. Hence, we denote the $n$-dimensional spherical distributions induced by $\Phi$ by $S_n(\Phi)$ and call $\Phi$ the characteristic generator.

**Remark.** The characteristic function of a spherical distribution is real valued, due to its symmetry.

Besides the above characterization of spherical distributions, Schoenberg [10] introduced the following stochastic representation, which we frequently use in the sequel. Denote by

$$\Omega_n = \{\Phi(\cdot) : \Phi(t_1^2 + \cdots + t_n^2) \text{ is an } n\text{-dimensional characteristic function}\}$$

the set of characteristic generators for $n$-dimensional spherical distributions.

**Theorem 2.2.** *Suppose that $X$ is spherically distributed with characteristic generator $\Phi \in \Omega_n$. Then $X$ has the representation*

$$X \overset{d}{=} R_n U^{(n)}, \tag{2.1}$$

*where the random variable $R_n \geq 0$ is independent of the n-dimensional random vector $U^{(n)}$ which is uniformly distributed on unit sphere in $\mathbb{R}^n$.*

The stochastic representation (2.1) will be a central issue of this paper, and therefore we state the proof for completeness.

*Proof.* We begin by proving the following equivalence. A function $\Phi$ is a characteristic generator, i.e. $\Phi \in \Omega_n$, if and only if

$$\Phi(x) = \int_0^\infty \Psi_n(xr^2) dF_n(r), \tag{2.2}$$

where $F_n : [0,\infty) \to [0,1]$ is a distribution function and

$$\Psi_n(y'y) = \int_{x'x=1} e^{iy'x} dS_n(x)/S_n \tag{2.3}$$

with $S_n(\cdot)$ denoting the uniform measure on the unit sphere in $\mathbb{R}^n$ and $S_n$ denoting the area of the unit sphere in $\mathbb{R}^n$. Observe that $\Psi_n(y'y)$ is the characteristic function of $U^{(n)}$.

i) Let $\Phi \in \Omega_n$ and $x$ be arbitrary with $x'x = \|x\|^2 = 1$; then $g(t_1, \ldots, t_n) := \Phi(t't) = \Phi\big((\|t\|x)'(\|t\|x)\big) = g(\|t\|x_1, \ldots, \|t\|x_n)$, where $g(t_1, \ldots, t_n)$ is the characteristic function of some random vector $Y$ with distribution function $H$. Then

$$\begin{aligned}
\Phi(t't) &= \frac{1}{S_n}\int_{\|x\|=1} g(\|t\|x_1, \ldots, \|t\|x_n)\, dS_n(x) \\
&= \frac{1}{S_n}\int_{\|x\|=1} \left[\int_{\mathbb{R}^n} e^{iy'(\|t\|x)} dH(y)\right] dS_n(x) \qquad (2.4)\\
&= \int_{\mathbb{R}^n} \left[\frac{1}{S_n}\int_{\|x\|=1} e^{i(\|t\|y)'x} dS_n(x)\right] dH(y) = \int_{\mathbb{R}^n} \Psi_n(\|t\|^2\|y\|^2)\, dH(y).
\end{aligned}$$

With $F_n(u) := \int_{\|y\|<u} dH(y)$ we obtain $\Phi(x) = \int_0^\infty \Psi_n(xu^2) dF_n(u)$.

ii) Let $\Phi(x)$ have representation (2.2) and let $R_n$ be a random variable with distribution function $F_n$ which is independent of $U^{(n)}$. Then

$$\begin{aligned}
\mathbb{E}(e^{it'R_nU^{(n)}}) &= \int_0^\infty \mathbb{E}(e^{irt'U^{(n)}})\, dF_n(r) \\
&= \int_0^\infty \Psi_n(\|t\|^2 r^2)\, dF_n(r) = \Phi(\|t\|^2) = \Phi(t't),
\end{aligned}$$

using the independence property of $U^{(n)}$ and $R_n$. Hence, $\Phi \in \Omega_n$.

iii) Combining part i) and ii), we obtain that

$$\mathbb{E}(e^{it'R_nU^{(n)}}) = \Phi(t't)$$

for some nonnegative random variable $R_n$ independent of $U^{(n)}$ and some function $\Phi$ if and only if $\Phi \in \Omega_n$. Finally the uniqueness theorem for characteristic functions yields the theorem. □

**Definition 2.3.** Let $X \in S_n(\Phi)$. Corresponding to the latter theorem we call $R_n$ the *generating random variable* or *generating variate*, and its distribution $F_n$ the *generating distribution function*. Further, we denote the univariate marginal distribution function of $X \in S_n(\Phi)$ by $G$.

Note that each margin of $X$ has an identical distribution function $G$, due to the symmetric form of the characteristic function of $X$. Before we go into details let us first state the main results for spherical distributions.

**Theorem 2.4.** *Let $X \in S_n(\Phi)$, $n \geq 2$, with stochastic representation $X \stackrel{d}{=} R_n U^{(n)}$.*
*$\alpha$) Suppose $F_n$, the distribution function of $R_n$, has a regularly varying tail. Then all bivariate margins have the tail dependence property.*
*$\beta$) If $X$ has a tail dependent bivariate margin, then the tail function of the univariate margins $\overline{G} = 1 - G$ is O-regularly varying.*
*$\gamma$) If $X$ has a tail dependent bivariate margin, then the tail function $\overline{F}_n = 1 - F_n$ of $R_n$ is O-regularly varying.*

**Remarks.**

1. Moreover, in the framework of Theorem 2.4 we prove that if $G$ has a regularly varying tail, i.e. $\overline{G} = 1 - G \in RV_{-\alpha}$ with $\alpha > 0$, and $x^\alpha \overline{F}_n(x)$ satisfies the Tauberian Condition 3.2 then all bivariate margins possess the tail dependence property.
2. Later we will state an equivalent theorem regarding densities, where we can prove a stronger result.

For the proof of Theorem 2.4 we need several lemmas and propositions, which yield interesting results in themselves. First we investigate the marginal characteristics of spherical distributions.

**Lemma 2.5.** *Let $X \in S_n(\Phi)$ and $(X^{(m)}, X^{(n-m)})'$ be a partition of $X \stackrel{d}{=} R_n U^{(n)}$ with $X^{(m)} \in I\!R^m$ and $X^{(n-m)} \in I\!R^{n-m}$. Then*
*$\alpha$)*

$$X^{(m)} \in S_m(\Phi)$$

*and*
*$\beta$)*

$$(X^{(m)}, X^{(n-m)})' \stackrel{d}{=} (R_n D_1 U^{(m)}, R_n D_2 U^{(n-m)})' \tag{2.5}$$

*holds, where $D_1^2$ is Beta$(\frac{m}{2}, \frac{n-m}{2})$ distributed and $D_2^2 = 1 - D_1^2$. Further $U^{(m)}$, $U^{(n-m)}$, and $(D_1^2, D_2^2)$ are mutually independent random vectors.*

For a proof we refer the reader to Fang, Kotz, and Ng [6].

**Remark.** Part $\alpha$) and the symmetry of the characteristic function of spherical distributions imply that all margins of $X$ are spherically distributed with the same characteristic generator.

The following lemma describes the relationship between the margins of a spherically distributed $X$ and its characteristic variate.

Let $X \in S_n(\Phi)$ and $X^{(m)} \in S_m(\Phi)$, $1 \leq m \leq n$, be the corresponding marginal random vector. Then $R_m \geq 0$, $1 \leq m \leq n$, denotes the random variable which relates to the stochastic representation

$$X^{(m)} \stackrel{d}{=} R_m U^{(m)}.$$

**Lemma 2.6.** *If $X \in S_n(\Phi)$ and $X^{(m)} \in S_m(\Phi)$, $1 \leq m \leq n$, have generating variates $R_n$ and $R_m$, respectively, then*

$$R_m \stackrel{d}{=} R_n B_m,$$

*where the random variable $B_m$ with $0 \le B_m \le 1$ is independent of $R_n$, and $B_m^2$ follows a Beta$(\frac{m}{2}, \frac{n-m}{2})$ distribution for $1 \le m < n$ and $B_n^2 \equiv 1$.*

*Proof.* This follows immediately by Lemma 2.5 setting $B_m := D_1$ and observing that $R_m = \|R_m U^{(m)}\| \overset{d}{=} \|R_n B_m U^{(m)}\| = R_n B_m$. □

## 3. Regular variation properties for spherical distributions

An immediate consequence of Lemma 2.6 for spherical distributions is that a generating variate $R_n$ with regularly varying tail implies a regularly varying tail of $R_m$, which is the generating variate of the corresponding $m$-dimensional margin.

**Proposition 3.1.** *Let $F_n$ and $F_m$ be the distribution functions of the generating variate $R_n$ and $R_m$ corresponding to $X \in S_n(\Phi)$ and $X^{(m)} \in S_m(\Phi)$, $1 \le m \le n$. If $F_n$ has a regularly varying tail, i.e., $\overline{F}_n \in RV_{-\alpha}$, $\alpha > 0$, then $\overline{F}_m$ is also regularly varying with the same index, i.e., $\overline{F}_m \in RV_{-\alpha}$.*

*Proof.* Recall that $\Phi \in \Omega_n$ if and only if

$$\Phi(x) = \int_0^\infty \Psi_n(xr^2) dF_n(r),$$

where $F_n : I\!R^+ \to [0,1]$ is a distribution function and
$\Psi_n(y'y) = \int_{x'x=1} e^{iy'x} dS_n(x)/S_n$ (see formula (2.3)). Let $\Phi \in \Omega_n$, then $\Phi \in \Omega_m$, $\forall 1 \le m \le n$. Consequently, for any fixed $m$, $1 \le m \le n$, there exists a distribution function $F_m$ such that

$$\Phi(x) = \int_0^\infty \Psi_m(xr^2) dF_m(r), \quad \text{and} \quad \Psi_m(y'y) = \int_{x'x=1} e^{iy'x} dS_m(x)/S_m.$$

i) Suppose $F_n$ has a regularly varying tail, i.e., $\overline{F}_n \in RV_{-\alpha}$, $\alpha > 0$. Applying Lemma 2.6 we obtain

$$\overline{F}_m(u) = \int_0^1 \overline{F}_n\Big(\frac{u}{y}\Big) dF_{B_m}(y), \tag{3.1}$$

where $F_{B_m}$ denotes the distribution function of $B_m$ and $B_m^2$ follows a $Beta(\frac{m}{2}, \frac{n-m}{2})$ distribution for $1 \le m < n$. Moreover, utilizing the uniform convergence Theorem 1.3 for regularly varying functions yields that for every $\varepsilon \in (0,\alpha]$, $\varepsilon < 1$, there exists a constant $K_\varepsilon \ge 0$ such that

$$\begin{aligned} \frac{\overline{F}_m(u)}{\overline{F}_n(u)} &= \int_0^1 \frac{\overline{F}_n(\frac{u}{y})}{\overline{F}_n(u)} dF_{B_m}(y) \le \int_0^1 \Big(\frac{1}{y}\Big)^{-\alpha+\varepsilon} (1+\varepsilon) dF_{B_m}(y) \\ &= (1+\varepsilon)\int_0^1 y^{\alpha-\varepsilon} dF_{B_m}(y) = (1+\varepsilon) I\!E B_m^{\alpha-\varepsilon} < \infty \end{aligned}$$

for all $u \ge K_\varepsilon$. Similarly, there exists $\hat{K}_\varepsilon \ge 0$ such that

$$\frac{\overline{F}_m(u)}{\overline{F}_n(u)} \ge (1-\varepsilon) I\!E B_m^{\alpha+\varepsilon} > 0$$

for all $u \ge \hat{K}_\varepsilon$. Because $\varepsilon > 0$ is arbitrary and $\lim_{\varepsilon\to 0} I\!E B^{\alpha\pm\varepsilon} = I\!E B^\alpha$, this implies

$$\lim_{u\to\infty} \frac{\overline{F}_m(u)}{\overline{F}_n(u)} = I\!E B^\alpha, \quad \alpha > 0.$$

Therefore, for all $t > 0$

$$\lim_{u\to\infty} \frac{\overline{F}_m(tu)}{\overline{F}_m(u)} = \lim_{u\to\infty} \frac{\overline{F}_m(tu)}{\overline{F}_n(tu)} \frac{\overline{F}_n(u)}{\overline{F}_m(u)} \frac{\overline{F}_n(tu)}{\overline{F}_n(u)} = t^{-\alpha}$$

holds and we conclude that $\overline{F}_m \in RV_{-\alpha},\ \alpha > 0,\ 1 \le m \le n.$

□

**Remark.** Similarly, one can show that the density function $f_m,\ 1 \le m \le n$, corresponding to $F_m$ is regularly varying with index $-\alpha - 1$, i.e., $f_m \in RV_{-\alpha-1},\ \alpha > 0$, if the density function $f_n$ of $F_n$ is regularly varying with the same index, i.e., $f_n \in RV_{-\alpha-1},\ \alpha > 0$.

The converse of Proposition 3.1 is also true under a further condition.

**Condition 3.2** (Tauberian condition, Bingham et.al. [2], p. 197). *Let $h : \mathbb{R}^+ \to \mathbb{R}$ be a measurable function, then $h$ satisfies the so-called Tauberian condition if*

$$\lim_{\lambda\to 1^+} \liminf_{x\to\infty} \inf_{t\in[1,\lambda]} \{h(tx) - h(x)\} \ge 0$$

*holds.*

**Remark.** The above condition is equivalent to $h$ being a slowly decreasing function (compare Bingham, et.al. [2], p.41).

**Proposition 3.3.** *Let $F_n$ and $F_m$ be the distribution functions of the generating variate $R_n$ and $R_m$ corresponding to $X \in S_n(\Phi)$ and $X^{(m)} \in S_m(\Phi)$, $1 \le m \le n$. If $F_m$ has a regularly varying tail for some $m \in \{1, \dots, n\}$, i.e., $\overline{F}_m \in RV_{-\alpha},\ \alpha > 0$, and $x^\alpha \overline{F}_n(x)$ satisfies the Tauberian Condition* 3.2 *then $\overline{F}_n$ is also regularly varying with the same index, i.e., $\overline{F}_n \in RV_{-\alpha}$.*

*Proof.* For this converse of Proposition 3.1 we need a deeper result. Assume that $F_m$ has a regularly varying tail, i.e., $\overline{F}_m \in RV_{-\alpha},\ \alpha > 0,\ 1 \le m < n$. Again, with Lemma 2.6 we obtain

$$\begin{aligned} \overline{F}_m(u) &= \int_0^1 \overline{F}_n\left(\frac{u}{y}\right) f_{B_m}(y)dy \\ &= \int_u^\infty \overline{F}_n(x) f_{B_m}\left(\frac{u}{x}\right)\frac{u}{x^2}dx \\ &= \int_0^\infty f_{B_m}\left(\frac{u}{x}\right)\frac{u}{x}\mathbf{1}_{(0,1)}\left(\frac{u}{x}\right)\overline{F}_n(x)\frac{dx}{x} = k \overset{M}{*} \overline{F}_n(u), \end{aligned}$$

where $f_{B_m}$ is the density function of $B_m$ and $k \overset{M}{*} \overline{F}_n$ is a Mellin convolution (see Definition 1.5) with kernel $k(t) = f_{B_m}(t)t\mathbf{1}_{(0,1)}(t),\ t > 0$. Because we want to transfer the regular variation property of $k \overset{M}{*} \overline{F}_n$ towards $\overline{F}_n$ we arrive at the Wiener-Tauberian Theorem for Mellin convolutions (see Bingham, Goldie, and Teugels [2], Theorem 4.8.3, p. 230). The latter theorem requires us to consider the Mellin transform of $k$, i.e.,

$$\check{k}(z) = \int_0^\infty t^{-z+1} f_{B_m}(t)\mathbf{1}_{(0,1)}(t)\frac{dt}{t}.$$

which converges for $-\infty < \text{Re}z < m$, since $B_m^2$ is $Beta\Big(\frac{m}{2}, \frac{n-m}{2}\Big)$ distributed, and therefore

$$f_{B_m}(t) = \frac{2}{Beta\big(\frac{m}{2}, \frac{n-m}{2}\big)} t^{m-1}(1-t^2)^{\frac{n-m}{2}-1}, \quad t > 0,$$

and

$$\check{k}(z) = \frac{2}{Beta\big(\frac{m}{2}, \frac{n-m}{2}\big)} \int_0^1 t^{-z+m-1}(1-t^2)^{\frac{n-m}{2}-1} dt$$

$$= \frac{Beta\big(\frac{-z+m}{2}, \frac{n-m}{2}\big)}{Beta\big(\frac{m}{2}, \frac{n-m}{2}\big)}. \tag{3.2}$$

The Wiener-Tauberian Theorem implies a regularly varying tail function $\overline{F}_n$, i.e. $\overline{F}_n \in RV_{-\alpha},\ \alpha > 0$, if the following two conditions hold:
i) the Wiener condition on the kernel: $\check{k} \neq 0$ for $\text{Re}(z) = -\alpha$, and
ii) the Tauberian condition:

$$\lim_{\lambda \to 1^+} \liminf_{x \to \infty} \inf_{t \in [1,\lambda]} \left\{ \frac{\overline{F}_n(tx)}{(tx)^{-\alpha}} - \frac{\overline{F}_n(x)}{x^{-\alpha}} \right\} \geq 0.$$

Since we assume that $x^\alpha \overline{F}_n$ fulfills the Tauberian condition it remains to show that the Wiener condition holds.
Recall from equation (3.2) that

$$\check{k}(z) = \frac{Beta\big(\frac{-z+m}{2}, \frac{n-m}{2}\big)}{Beta\big(\frac{m}{2}, \frac{n-m}{2}\big)}, \ -\infty < \text{Re}(z) < m,\ 1 \leq m < n,$$

and therefore $\check{k}(z) > 0$, since $\Gamma(\omega) > 0$ for $\text{Re}(\omega) > 0$, according to Freitag and Busan [7], Theorem 1.10, p. 199. □

**Remark.** We have not shown yet, whether one can drop the Tauberian condition in the latter theorem. This issue will be addressed in a forthcoming work.

**Proposition 3.4.** *Let $X \stackrel{d}{=} R_n U^{(n)} \in S_n(\Phi)$. Suppose the distribution function of $R_n$ has a regularly varying tail, i.e., $\overline{F}_n \in RV_{-\alpha},\ \alpha > 0$. Then the tail function $\overline{G}$ of the univariate margins of $X$ is also regularly varying with the same index.*

*Proof.* Due to the symmetry of spherical distributions we may write $G$ as the univariate distribution function of $X \in S_n(\Phi)$. Recall that the univariate margin $X^{(1)}$ possesses the stochastic representation $X^{(1)} \stackrel{d}{=} R_1 U^{(1)}$, where $U^{(1)}$ is independent of $R_1$ and $U^{(1)}$ is Bernoulli distributed. Then for all $t > 0$

$$\lim_{x \to \infty} \frac{\overline{G}(tx)}{\overline{G}(x)} = \lim_{x \to \infty} \frac{\mathbb{P}(X^{(1)} > tx)}{\mathbb{P}(X^{(1)} > x)}$$

$$= \lim_{x \to \infty} \frac{\mathbb{P}(R_1 U^{(1)} > tx)}{\mathbb{P}(R_1 U^{(1)} > x)} = \lim_{x \to \infty} \frac{\mathbb{P}(R_1 > tx)}{\mathbb{P}(R_1 > x)} = t^{-\alpha},$$

because $R_1$ inherits the regular variation property from $R_n$ by Proposition 3.1. The second to last equality follows from the independence of $U^{(1)}$ and $R_1 \geq 0$, and the Bernoulli distribution of $U^{(1)}$. □

It is obvious from the definition of spherical distributions that if $X \in S_n(\Phi)$ possesses a density, it must be of the form $g(t't)$, for some measurable function $g : \mathbb{R}^+ \to \mathbb{R}^+$.

**Definition 3.5.** Suppose $X \in S_n(\Phi)$ possesses a density function $g(t't)$. Then $g$ is called the *density generator* of $X$ and we write $X \in S_n(g)$.

The next lemma connects the density generator of a spherical distribution with the density of its generating variate.

**Lemma 3.6.** *Let $X \in S_n(\Phi)$ with stochastic representation $X \stackrel{d}{=} R_n U^{(n)}$. Then $X$ possesses a density generator $g$ if and only if $R_n$ has a density $f_n$. The relationship between $f_n$ and $g$ is given by*

$$f_n(x) = \frac{2\pi^{n/2}}{\Gamma(n/2)} x^{n-1} g(x^2), \qquad x \geq 0. \tag{3.3}$$

*Proof.* First, we assume that $X$ has a density generator $g$. We apply the Liouville/Dirichlet integral formula (see e.g. Whittaker and Watson [11], Ch. 12.5) which holds for any nonnegative measurable function $v$ :

$$\int_{\mathbb{R}^n} v\Big(\sum_{i=1}^{n} x_i^2\Big) dx_1 \ldots dx_n = \frac{\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty y^{n/2-1} v(y) dy.$$

Thus, for every measurable function $h \geq 0$ we deduce

$$\begin{aligned} \mathbb{E}h(R_n) = \mathbb{E}h(\|X\|) &= \int_{\mathbb{R}^n} h(\sqrt{x'x}) g(x'x) dx \\ &= \frac{\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty h(\sqrt{y}) g(y) y^{n/2-1} dy = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty h(r) g(r^2) r^{n-1} dr. \end{aligned}$$

Setting $h(r) = \mathbf{1}_{[0,x]}(r)$ for $x \geq 0$, yields equation (3.3). For the converse we assume that $R_n$ has density $f_n$ and $X \in S_n(\Phi)$ with stochastic representation $X \stackrel{d}{=} R_n U^{(n)}$.Then

$$\begin{aligned} \mathbb{E}e^{it'X} &= \int_0^\infty \mathbb{E}e^{i(rt)'U^{(n)}} f_n(r) dr \\ &= \frac{\Gamma(n/2)}{\pi^{n/2}} \int_0^\infty \frac{\pi^{n/2}}{2\Gamma(n/2) u^{\frac{n-1}{2}}} u^{n/2-1} \mathbb{E}e^{i(\sqrt{u}t)'U^{(n)}} f_n(\sqrt{u}) du \\ &= \frac{\Gamma(n/2)}{2\pi^{n/2}} \int_{\mathbb{R}^n} \mathbb{E}e^{i(\|x\|t)'U^{(n)}} \frac{f_n(\|x\|)}{\|x\|^{n-1}} dx_1 \ldots dx_n \\ &= \int_{\mathbb{R}^n} \Psi_n(\|x\|^2 \|t\|^2) \frac{\Gamma(n/2)}{2\pi^{n/2}} \frac{f_n(\|x\|)}{\|x\|^{n-1}} dx_1 \ldots dx_n \\ &= \int_{\mathbb{R}^n} e^{it'x} \frac{\Gamma(n/2)}{2\pi^{n/2}} \frac{f_n(\|x\|)}{\|x\|^{n-1}} dx_1 \ldots dx_n = \Phi(t't), \end{aligned}$$

according to equation (2.4) and to the fact that $\int_{\mathbb{R}^n} \frac{\Gamma(n/2)}{2\pi^{n/2}} \frac{f_n(\|x\|)}{\|x\|^{n-1}} dx_1 \ldots dx_n = 1$. The uniqueness theorem for characteristic functions leads to the conclusion. □

**Proposition 3.7.** *Let $X \in S_n(\Phi)$ with stochastic representation $X \stackrel{d}{=} R_n U^{(n)}$. Then $X$ has a regularly varying density generator $g \in RV_{-(\alpha+n)/2}$, $\alpha > 0$, if and only if $R_n$ possesses a regularly varying density $f_n \in RV_{-\alpha-1}$, $\alpha > 0$.*

*Proof.* The proposition follows immediately by Lemma 3.6 with

$$\lim_{x\to\infty} \frac{f_n(xt)}{f_n(x)} = \lim_{x\to\infty} t^{n-1} \frac{g(x^2t^2)}{g(x^2)} = t^{-\alpha-1}.$$

□

Another consequence of Lemma 2.6 concerning O-regularly varying tail functions is stated in the next proposition.

**Proposition 3.8.** *Let $X \in S_n(\Phi)$. Suppose that the generating variate $R_{m_0}$, $1 \leq m_0 \leq n$ of the margin $X_{m_0}$ possesses an O-regularly varying tail, i.e., $\overline{F}_{m_0} \in OR$. Then $\overline{F}_m \in OR$ for all $1 \leq m \leq n$, where $F_m$ denotes the distribution function of the generating random variable $R_m$.*

*Proof.* Suppose $X_{m_0} \in S_{m_0}(\Phi)$ and $\overline{F}_{m_0} \in OR$.
i) Consider first the case $1 \leq m < m_0$. Recall from Lemma 2.6 that $R_m \overset{d}{=} R_{m_0} B_m$, where $R_{m_0}$ and $B_m$ are independent and $B_m^2$ is *Beta*$(m/2, (m_0 - m)/2)$ distributed. Then for all $t > 1$

$$\begin{aligned}
\liminf_{x\to\infty} \frac{\overline{F}_m(tx)}{\overline{F}_m(x)} &= \liminf_{x\to\infty} \frac{\mathbb{P}(R_m > tx)}{\mathbb{P}(R_m > x)} = \liminf_{x\to\infty} \frac{\mathbb{P}(R_m > tx)}{\mathbb{P}(R_{m_0} B_m > x)} \\
&\geq \liminf_{x\to\infty} \frac{\mathbb{P}(R_m > tx)}{\mathbb{P}(R_{m_0} > x)} \geq \liminf_{x\to\infty} \frac{\mathbb{P}(R_{m_0} > tx)}{\mathbb{P}(R_{m_0} > x)} \cdot \liminf_{x\to\infty} \frac{\mathbb{P}(R_m > tx)}{\mathbb{P}(R_{m_0} > tx)}.
\end{aligned}$$

Further,

$$\begin{aligned}
\liminf_{x\to\infty} \frac{\mathbb{P}(R_m > tx)}{\mathbb{P}(R_{m_0} > tx)} &= \liminf_{x\to\infty} \frac{\mathbb{P}(R_{m_0} B_m > tx)}{\mathbb{P}(R_{m_0} > tx)} \\
&= \liminf_{x\to\infty} \int_0^1 \frac{\mathbb{P}(R_{m_0} > tx/b)}{\mathbb{P}(R_{m_0} > tx)} dF_{B_m}(b) \\
&\geq \int_0^1 \liminf_{x\to\infty} \frac{\mathbb{P}(R_{m_0} > tx/b)}{\mathbb{P}(R_{m_0} > tx)} dF_{B_m}(b) > 0,
\end{aligned}$$

since $\liminf_{x\to\infty} \frac{\mathbb{P}(R_{m_0} > tx/b)}{\mathbb{P}(R_{m_0} > tx)} > 0$ for all $0 < b \leq 1$. The last but one inequality follows from Fatou's lemma. Combining these inequalities and applying the prerequisite $\overline{F}_{m_0} \in OR$ yields the desired result.
ii) Let now $m_0 < m \leq n$. Again from Lemma 2.6 we know that $R_{m_0} \overset{d}{=} R_m B_{m_0}$, where $R_m$ and $B_{m_0}$ are independent and $B_{m_0}^2$ is *Beta*$(m_0/2, (m - m_0)/2)$ distributed. Then for all $t > 1$

$$\begin{aligned}
0 < \liminf_{x\to\infty} \frac{\overline{F}_{m_0}(tx)}{\overline{F}_{m_0}(x)} &= \liminf_{x\to\infty} \frac{\mathbb{P}(R_m B_{m_0} > tx)}{\mathbb{P}(R_m B_{m_0} > x)} \\
&\leq \liminf_{x\to\infty} \frac{\mathbb{P}(R_m > tx)}{\mathbb{P}(R_m B_{m_0} > x)} = \liminf_{x\to\infty} \Big( \int_0^1 \frac{\mathbb{P}(bR_m > x)}{\mathbb{P}(R_m > tx)} dF_{B_{m_0}}(b) \Big)^{-1} \\
&= \Big( \limsup_{x\to\infty} \int_0^1 \frac{\mathbb{P}(R_m > \frac{x}{bt})}{\mathbb{P}(R_m > x)} dF_{B_{m_0}}(b) \Big)^{-1}
\end{aligned}$$

where $B_{m_0}^2$ is *Beta*$(m_0/2, (m - m_0)/2)$ distributed and independent of $R_m$. Therefore,

$$\limsup_{x\to\infty} \int_0^1 \frac{\mathbb{P}(R_m > \frac{x}{bt})}{\mathbb{P}(R_m > x)} dF_{B_{m_0}}(b) < \infty.$$

Assume there exists some $b_0$ with $1 \geq b_0 > 1/t$ such that $\limsup_{x\to\infty} \frac{\mathbb{P}(R_m > \frac{x}{b_0 t})}{\mathbb{P}(R_m > x)} = \infty$. Then

$$\limsup_{x\to\infty} \frac{\mathbb{P}(R_m > \frac{x}{bt})}{\mathbb{P}(R_m > x)} \geq \limsup_{x\to\infty} \frac{\mathbb{P}(R_m > \frac{x}{b_0 t})}{\mathbb{P}(R_m > x)} = \infty$$

for all $b \in [b_0, 1]$. Moreover, for all $N \in \mathbb{N}$, there exists $x_N \geq 0$ such that

$$\frac{\mathbb{P}(R_m > \frac{x_N}{bt})}{\mathbb{P}(R_m > x_N)} \geq N \qquad \forall\, b \in [b_0, 1]$$

and thus

$$\int_0^1 \frac{\mathbb{P}(R_m > \frac{x_N}{bt})}{\mathbb{P}(R_m > x_N)} dF_{B_{m_0}}(b) \geq N(1 - F_{B_{m_0}}(b_0)).$$

Because $N$ can be chosen arbitrarily large we conclude

$$\limsup_{x\to\infty} \int_0^1 \frac{\mathbb{P}(R_m > \frac{x}{bt})}{\mathbb{P}(R_m > x)} dF_{B_m}(b) = \infty,$$

which leads to a contradiction. Hence,

$$\limsup_{x\to\infty} \frac{\mathbb{P}(R_m > \frac{x}{bt})}{\mathbb{P}(R_m > x)} < \infty \qquad \forall\, b \in [\frac{1}{t}, 1]$$

or equivalently

$$\liminf_{x\to\infty} \frac{\mathbb{P}(R_m > tbx)}{\mathbb{P}(R_m > x)} > 0 \qquad \forall\, b \in [\frac{1}{t}, 1] \text{ and } t > 1.$$

This completes the proof. □

## 4. Tail dependence for spherical distributions

We are ready to prove Theorem 2.4. For completeness we state the theorem again.

**Theorem 2.4.** *Let $X \in S_n(\Phi)$, $n \geq 2$, with stochastic representation $X \stackrel{d}{=} R_n U^{(n)}$.*
*$\alpha$) Suppose $F_n$, the distribution function of $R_n$, has a regularly varying tail. Then all bivariate margins have the tail dependence property.*
*$\beta$) If $X$ has a tail dependent bivariate margin, then the tail function of the univariate margins $\overline{G}$ is O-regularly varying.*
*$\gamma$) If $X$ has a tail dependent bivariate margin, then the tail function $\overline{F}_n$ of $R_n$ is O-regularly varying.*

*Proof.* $\alpha$) All bivariate margins of $X \in S_n(\Phi)$ possess the same tail dependence index due to the symmetry of the characteristic function $\Phi(t't)$ of $X$. Thus, it suffices to prove tail dependence for the bivariate margin $X^{(2)} = (X_1, X_2)$. Again, due to the symmetry of the characteristic function, both univariate margins possess the same marginal distribution function $G$. Let $F_n$, the distribution function of $R_n$, have a regularly varying tail $\overline{F}_n \in RV_{-\alpha}$, $\alpha > 0$, i.e.

$$\lim_{x\to\infty} \frac{\overline{F}_n(tx)}{\overline{F}_n(x)} = t^{-\alpha}, \quad \forall t > 0. \tag{4.1}$$

Then Proposition 3.1 requires also a regularly varying tail for $F_2$, where $F_2$ denotes the distribution function of $R_2$ corresponding to $X^{(2)} \stackrel{d}{=} R_2 U^{(2)}$. Note that regular variation (at $\infty$) implies that $F_2$ has an infinite right endpoint. Recall from

Lemma 2.5 that $X^{(2)}$ possesses the stochastic representation $X^{(2)} = (X_1, X_2)' = (R_2 D U_1^{(1)}, R_2\sqrt{1-D^2}U_2^{(1)})'$, where $U_1^{(1)}$, $U_2^{(1)}$, $R_2$, and $D$ are mutually independent, $U_1^{(1)}$, $U_2^{(1)}$ are Bernoulli distributed, and $D^2$ is $Beta(\frac{1}{2}, \frac{1}{2})$ distributed. We denote the distribution function of $D$ by $F_D$. Then

$$\begin{aligned}
&\lim_{v\to 1^-} \mathbb{P}(X_1 > G^{-1}(v)|X_2 > G^{-1}(v)) = \lim_{x\to\infty} \mathbb{P}(X_1 > x|X_2 > x) \\
&\quad = \lim_{x\to\infty} \mathbb{P}(R_2\sqrt{1-D^2}U_1^{(1)} > x|R_2 D U_2^{(1)} > x) \\
&\quad = \lim_{x\to\infty} \frac{1}{2}\mathbb{P}(R_2\sqrt{1-D^2} > x|R_2 D > x) \\
&\quad = \lim_{x\to\infty} \frac{1}{2}\frac{\int_0^1 \mathbb{P}\Big(R > \max(\frac{1}{\sqrt{1-u^2}}, \frac{1}{u})x\Big)dF_D(u)}{\int_0^1 \mathbb{P}\Big(R > \frac{x}{u}\Big)dF_D(u)} \\
&\quad = \lim_{x\to\infty} \frac{1}{2}\frac{\int_{1/\sqrt{2}}^1 \overline{F}_2\Big(\frac{x}{\sqrt{1-u^2}}\Big)/\overline{F}_2(x)dF_D(u) + \int_0^{1/\sqrt{2}} \overline{F}_2\Big(\frac{x}{u}\Big)/\overline{F}_2(x)dF_D(u)}{\int_0^1 \overline{F}_2\Big(\frac{x}{u}\Big)/\overline{F}_2(x)dF_D(u)}.
\end{aligned}$$

Together with the uniform convergence Theorem 1.3 we infer that for any $\varepsilon > 0$ exists a constant $K_\varepsilon > 0$ such that for all $x \geq K_\varepsilon$

$$\mathbb{P}(X_1 > x|X_2 > x) \leq \left(\frac{1+\varepsilon}{1-\varepsilon}\right)\frac{1}{2}\frac{\int_{1/\sqrt{2}}^1(\sqrt{1-u^2})^\alpha dF_D(u) + \int_0^{1/\sqrt{2}} u^\alpha dF_D(u)}{\int_0^1 u^\alpha dF_D(u)}.$$

Analogously, we obtain

$$\mathbb{P}(X_1 > x|X_2 > x) \geq \left(\frac{1-\varepsilon}{1+\varepsilon}\right)\frac{1}{2}\frac{\int_{1/\sqrt{2}}^1(\sqrt{1-u^2})^\alpha dF_D(u) + \int_0^{1/\sqrt{2}} u^\alpha dF_D(u)}{\int_0^1 u^\alpha dF_D(u)}.$$

Because $\varepsilon > 0$ can be chosen arbitrarily small we finally conclude for $\alpha > 0$

$$\begin{aligned}
&\lim_{v\to 1^-} \mathbb{P}(X_1 > G^{-1}(v)|X_2 > G^{-1}(v)) \\
&\qquad = \frac{1}{2}\frac{\int_{1/\sqrt{2}}^1(\sqrt{1-u^2})^\alpha dF_D(u) + \int_0^{1/\sqrt{2}} u^\alpha dF_D(u)}{\int_0^1 u^\alpha dF_D(u)} = \lambda \in (0,1]
\end{aligned}$$

and consequently $X$ possesses the tail dependence property.

Using the fact that $D^2$ is $Beta(\frac{1}{2}, \frac{1}{2})$ distributed and therefore $D$ has density

$$f_D(u) = \frac{2(1-u^2)^{-\frac{1}{2}}}{Beta(\frac{1}{2}, \frac{1}{2})},$$

we additionally obtain the tail dependence index $\lambda$ by

$$\lambda = \lim_{x\to\infty} \frac{\int_0^{1/\sqrt{2}} \overline{F}_2(x/u)\frac{1}{\sqrt{1-u^2}}du}{\int_0^1 \overline{F}_2(x/u)\frac{1}{\sqrt{1-u^2}}du} = \frac{\int_0^{1/\sqrt{2}} \frac{u^\alpha}{\sqrt{1-u^2}}du}{\int_0^1 \frac{u^\alpha}{\sqrt{1-u^2}}du}.$$

Before proving part $\beta$) we first prove part $\gamma$).

$\gamma$) Assume that $X \in S_n(\Phi)$ possesses a bivariate margin with positive tail dependence coefficient $\lambda$. Hence, all bivariate margins are tail dependent with equal tail dependence coefficient, using the same argumentation as in part $\alpha$). Without loss of generality we consider the bivariate margin $X^{(2)} = (X_1, X_2)$. Employing the same notation as in part $\alpha$) tail dependence is equivalent to the existence of the following limit

$$\frac{\int_0^{1/\sqrt{2}} \overline{F}_2(x/u)\frac{1}{\sqrt{1-u^2}}du}{\int_0^1 \overline{F}_2(x/u)\frac{1}{\sqrt{1-u^2}}du} \to \lambda \in (0,1] \quad (x \to \infty).$$

Hence, there exist constants $\varepsilon > 0$ and $K_\varepsilon \geq 0$ such that $\lambda - \varepsilon > 0$ and for all $x \geq K_\varepsilon$

$$\begin{aligned}\frac{\pi}{4}\overline{F}_2(\sqrt{2}x) &\geq \int_0^{1/\sqrt{2}} \overline{F}_2(x/u)\frac{1}{\sqrt{1-u^2}}du \\ &\geq (\lambda-\varepsilon)\int_0^1 \overline{F}_2(x/u)\frac{1}{\sqrt{1-u^2}}du \\ &\geq (\lambda-\varepsilon)\int_0^1 \overline{F}_2(x/u)du \geq (\lambda-\varepsilon)\overline{F}_2(\sqrt[3]{2}x)(1-1/\sqrt[3]{2}).\end{aligned}$$

These inequalities lead to

$$\frac{\overline{F}_2(\sqrt{2}x)}{\overline{F}_2(\sqrt[3]{2}x)} \geq \frac{4(1-1/\sqrt[3]{2})(\lambda-\varepsilon)}{\pi} =: \hat{c} > 0$$

for all $x \geq K_\varepsilon$. The latter is equivalent with characterizing $\overline{F}_2$ as O-regularly varying, since $\overline{F}_2$ is monotone decreasing (see also Bingham, Goldie, and Teugels [2], p.65, Corollary 2.0.6). Finally, Proposition 3.8 implies an O-regularly varying tail function $\overline{F}$.

$\beta$) Suppose again that $X \in S_n(\Phi)$ possesses a bivariate tail dependent margin. Then part $\gamma$) and Proposition 3.8 yield

$$\liminf_{x\to\infty} \frac{\overline{G}(tx)}{\overline{G}(x)} = \liminf_{x\to\infty} \frac{\mathbb{P}(R_1U^{(1)} > tx)}{\mathbb{P}(R_1U^{(1)} > x)} = \liminf_{x\to\infty} \frac{\overline{F}_1(tx)}{\overline{F}_1(x)} > 0,$$

where $R_1U^{(1)}$ denotes the stochastic representation of $X^{(1)}$. Therefore, $\overline{G}$ is O-regularly varying. This completes the proof. □

**Corollary 4.1.** *Let $X \in S_n(\Phi)$, $n \geq 2$, with stochastic representation $X \stackrel{d}{=} R_nU^{(n)}$. If $G$ has a regularly varying tail, i.e. $\overline{G} \in RV_{-\alpha}$, $\alpha > 0$, and $x^\alpha\overline{F}_n(x)$ satisfies the Tauberian Condition 3.2, where $\overline{F}_n$ denotes the tail function of the generating variate $R_n$, then all bivariate margins possess the tail dependence property.*

*Proof.* Let $X \in S_n(\Phi)$. Suppose the corresponding one-dimensional distribution function $G$ has a regularly varying tail with index $-\alpha$, $\alpha > 0$. Recall that the univariate margin $X^{(1)}$ possesses the stochastic representation $X^{(1)} \stackrel{d}{=} R_1U^{(1)}$, where $U^{(1)}$ is independent of $R_1$ and $U^{(1)}$ is Bernoulli distributed. Then for all $t > 0$

$$\begin{aligned}\lim_{x\to\infty}\frac{\overline{G}(tx)}{\overline{G}(x)} &= \lim_{x\to\infty}\frac{\mathbb{P}(X^{(1)} > tx)}{\mathbb{P}(X^{(1)} > x)} \\ &= \lim_{x\to\infty}\frac{\mathbb{P}(R_1U^{(1)} > tx)}{\mathbb{P}(R_1U^{(1)} > x)} = \lim_{x\to\infty}\frac{\mathbb{P}(R_1 > tx)}{\mathbb{P}(R_1 > x)} = t^{-\alpha}.\end{aligned}$$

Therefore, $R_1$ has also a regularly varying tail $\overline{F}_1$ with index $-\alpha$, and consequently the tail function $\overline{F}_n$ inherits the same property according to Proposition 3.3. Making use of Theorem 2.4 part $\alpha$) yields the desired conclusion. □

**Remark.** We have not shown yet, whether regular variation of the generating variate is equivalent to the tail dependence property for spherically distributed random vectors. To answer this open question one has to consider the *ratio Mercerian theorem* discussed in Bingham and Inoue [3]. The ratio Mercerian theorem asserts that under adequate conditions

$$\lim_{x\to\infty} \frac{\int_0^{1/\sqrt{2}} \overline{F}_2(x/u)\frac{1}{\sqrt{1-u^2}}du}{\int_0^1 \overline{F}_2(x/u)\frac{1}{\sqrt{1-u^2}}du} = \lambda \in (0,1]$$

implies $\overline{F}_2 \in RV_{-\alpha},\ \alpha > 0$. One of the key assumptions is that $-\alpha$ is the only zero of

$$\int_0^1 \frac{t^\alpha}{\sqrt{1-t^2}}dt \int_0^{1/\sqrt{2}} \frac{t^z}{\sqrt{1-t^2}}dt - \int_0^{1/\sqrt{2}} \frac{t^\alpha}{\sqrt{1-t^2}}dt \int_0^1 \frac{t^z}{\sqrt{1-t^2}}dt$$

in some vertical strip $a \le \mathrm{Re}(z) \le b$ such that $-\alpha \in (a,b)$.

In Section 6 we encounter spherical distributions which are given by their density functions. Thus, one should have a similar result to Theorem 2.4 regarding density functions of spherically distributed random vectors. First we prove a necessary lemma.

**Lemma 4.2.** *Let $F$ be the distribution function of an absolutely continuous non-negative random variable such that its tail function $\overline{F} \in OR$ and the corresponding density function $f$ is eventually decreasing. Further if*

$$\limsup_{x\to\infty} \frac{\overline{F}(bx)}{\overline{F}(x)} < 1 \tag{4.2}$$

*for some $b > 1$, then $f \in OR$ holds.*

*Proof.* We show first: There exist some constants $K, c_1, c_2 > 0$ such that

$$0 < c_1 \le \frac{xf(x)}{\overline{F}(x)} \le c_2 < \infty \qquad \forall\, x \ge K. \tag{4.3}$$

i) Let $\overline{F} \in OR$, then there exist $K, c_2 > 0$ such that $\overline{F}(2x)/\overline{F}(x) \ge \frac{2}{c_2} > 0$ and $f$ is decreasing for all $x \ge K/2$. Further

$$1 \ge \frac{\overline{F}(x) - \overline{F}(2x)}{\overline{F}(x)} = \frac{\int_x^{2x} f(u)du}{\overline{F}(x)} \ge \frac{xf(2x)}{\overline{F}(x)}$$

for all $x \ge K/2$ and therefore

$$\frac{2xf(2x)}{\overline{F}(2x)} \le \frac{xf(2x)}{\overline{F}(x)}\frac{2\overline{F}(x)}{\overline{F}(2x)} \le \frac{2\overline{F}(x)}{\overline{F}(2x)} \le c_2$$

for all $x \ge K/2$. Thus, $\dfrac{xf(x)}{\overline{F}(x)} \le c_2 < \infty$ for all $x \ge K$.

ii) Let $\overline{F} \in OR$, with $\limsup_{x\to\infty} \frac{\overline{F}(bx)}{\overline{F}(x)} < 1$ for some $b > 1$, then there exist $\varepsilon, K > 0$ such that $\frac{\overline{F}(bx)}{\overline{F}(x)} \le 1 - \varepsilon$ for all $x \ge K$. Further

$$0 < \varepsilon \le 1 - \frac{\overline{F}(bx)}{\overline{F}(x)} = \frac{\int_x^{bx} f(u)du}{\overline{F}(x)} \le \frac{xf(x)(b-1)}{\overline{F}(x)}$$

and therefore

$$0 < c_1 := \frac{\varepsilon}{b-1} \leq \frac{xf(x)}{\overline{F}(x)}$$

for all $x \geq K$.

iii) The conclusion $f \in OR$ follows now immediately by

$$0 < \liminf_{x\to\infty} \frac{c_1\overline{F}(tx)}{tc_2\overline{F}(x)} \leq \liminf_{x\to\infty} \frac{f(tx)}{f(x)} \leq \limsup_{x\to\infty} \frac{f(tx)}{f(x)} \leq 1.$$

□

The next condition seems to be more appropriate and easier to check in the context of density generators than (4.2).

**Condition 4.3.** *Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a measurable function eventually decreasing such that for some $\varepsilon > 0$*

$$\limsup_{x\to\infty} \frac{h(tx)}{h(x)} \leq 1 - \varepsilon \quad \textit{uniformly} \, \forall \, t > 1.$$

**Theorem 4.4.** *Let $X \in S_n(g)$, $n \geq 2$, be absolutely continuous with density generator $g$.*
*$\alpha$) Suppose $g$ is a regularly varying function, i.e., $g \in RV_{-(\alpha+n)/2}$ with $\alpha > 0$, then all bivariate margins of $X$ possess the tail dependence property.*
*$\beta$) If $X$ has a tail dependent bivariate margin and $g$ satisfies Condition 4.3, then $g$ must be O-regularly varying.*

*Proof.* $\alpha$) Let $g$ be the density generator of $X \in S_n(g)$, which is supposed to be regularly varying with index $-(\alpha+n)/2$, $\alpha > 0$. Then $f_n$ is also regularly varying with index $-\alpha-1$, $\alpha > 0$, according to Proposition 3.7. Consequently, Karamata's Theorem (see Bingham, Goldie, and Teugels [2], p. 26) implies that $F_n$ is regularly varying with index $-\alpha$, $\alpha > 0$, i.e. $F_n \in RV_{-\alpha}$. The conclusion follows by Theorem 2.4.
$\beta$) Suppose $X \in S_n(g)$ possesses a tail dependent bivariate margin. Recall, in that case all bivariate margins are tail dependent with the same tail dependence coefficient. According to Theorem 2.4 the distribution function of the univariate margins $G$ must be O-regularly varying. Further $g$ satisfies Condition 4.3 and the density function $g(x^2)$ of $G$ inherits this condition, which requires (4.2) for $G$. Therefore, Lemma 4.2 implies that $g$ is O-regularly varying. □

**Remark.** The tail dependence coefficient $\lambda$ for spherical distributions possessing regularly varying generating variates with index $-\alpha$, $\alpha > 0$, or regularly varying density generator with index $-\alpha/2 - 1$, $\alpha > 0$, is given by

$$\lambda = \frac{\displaystyle\int_0^{1/\sqrt{2}} \frac{u^\alpha}{\sqrt{1-u^2}} du}{\displaystyle\int_0^1 \frac{u^\alpha}{\sqrt{1-u^2}} du}. \tag{4.4}$$

See also Figure 1 below.

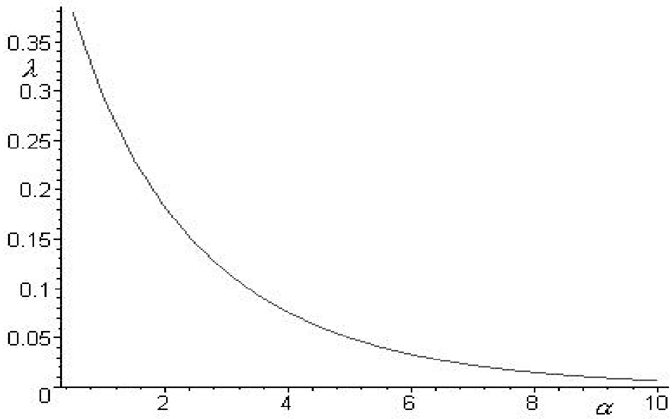


FIGURE 1. Tail dependence index $\lambda$ versus regular variation index $\alpha$

## 5. Tail dependence for elliptically contoured distributions

After investigating the relationship of spherical distributions and tail dependence in the last section we turn now to the more general case of elliptically distributed random vectors.

**Definition 5.1** (Elliptically contoured distribution)**.** Let $X$ be an $n$-dimensional random vector. Then $X$ is called elliptically distributed with parameters $\mu \in \mathbb{R}^n$ and $\Sigma \in \mathbb{R}^{n\times n}$ if

$$X \stackrel{d}{=} \mu + A'Y, \tag{5.1}$$

where $Y$ is a $m$-dimensional spherically distributed random vector, i.e., $Y \in S_m(\Phi)$, $A \in \mathbb{R}^{m\times n}$ with $A'A = \Sigma$, and rank($\Sigma$)=$m$. We denote by $E_n(\mu, \Sigma, \Phi)$ the family of $n$-dimensional elliptically distributed random vectors with parameters $\mu$, $\Sigma$, and characteristic generator $\Phi$.

According to the stochastic representation of spherical distributions we may represent every n-dimensional elliptically distributed random vector $X \in E_n(\mu, \Sigma, \Phi)$ with parameters $\mu$ and positive semidefinite matrix $\Sigma$, $\text{rank}(\Sigma) = m,\ m \le n$ by

$$X \stackrel{d}{=} \mu + R_m A' U^{(m)},$$

where $A'A = \Sigma$ and the random variable $R_m \ge 0$ is independent of the $m$-dimensional random vector $U^{(m)}$. The random vector $U^{(m)}$ is uniformly distributed on the unit sphere in $\mathbb{R}^m$. As in Section 2 we denote by $F_m$ the distribution function of $R_m$.

Following along the lines of the previous section we first state the main result of this section before we go into details.

**Theorem 5.2.** *Let $X \in E_n(\mu, \Sigma, \Phi)$, $n \ge 2$, with positive definite matrix $\Sigma$ and stochastic representation $X \stackrel{d}{=} \mu + A'Y \stackrel{d}{=} \mu + R_n A' U^{(n)}$.*
*$\alpha$) If $X$ has a tail dependent bivariate margin, then the tail function $\overline{F}_n$ must be O-regularly varying.*
*$\beta$) If $X$ has a tail dependent bivariate margin, then the tail function $\overline{G}$ must be O-regularly varying, where $G$ denotes the distribution function of the univariate*

*margins of Y.*
*γ) Suppose the distribution function $F_n$ of $R_n$ has a regularly varying tail. Then all bivariate margins are tail dependent.*
*δ) Suppose the distribution function G has a regularly varying tail, i.e. $\overline{G} \in RV_{-\alpha}$, $\alpha > 0$, and $x^\alpha \overline{F}_n(x)$ satisfies the Tauberian Condition 3.2 then all bivariate margins possess the tail dependence property.*

Before proving the theorem we present two preliminary steps. The following lemma is the equivalence to Lemma 2.6 for spherical distributions.

**Lemma 5.3.** *Suppose $X \in E_n(\mu, \Sigma, \Phi)$ with stochastic representation $X \overset{d}{=} \mu + R_m A' U^{(m)}$ and $X \overset{d}{=} \hat{\mu} + R_{\hat{m}} \hat{A}' U^{(\hat{m})}$ with $n \geq m \geq \hat{m}$. Then there exists a constant $c > 0$ such that*

$$\mu = \hat{\mu}, \quad \hat{A}'\hat{A} = cA'A, \quad R_{\hat{m}} \overset{d}{=} \frac{1}{\sqrt{c}} B_{\hat{m}} R_m,$$

*where $R_m$ is independent of $B_{\hat{m}}$ and $B_{\hat{m}}^2$ follows a Beta$(\frac{\hat{m}}{2}, \frac{m-\hat{m}}{2})$ distribution if $m > \hat{m}$ and $B_{\hat{m}} \equiv 1$ if $m = \hat{m}$.*

For a proof we refer the reader to Cambanis, Huang, and Simons [4], pp. 372.

**Lemma 5.4.** *Let $X \in E_n(\mu, \Sigma, \Phi)$ with stochastic representation $X \overset{d}{=} \mu + A'Y$, $Y \in S_m(\Phi)$, $m \leq n$ $A \in \mathbb{R}^{m \times n}$, and $A'A = \Sigma$. If $\hat{A} \in \mathbb{R}^{m \times n}$ with $\hat{A}'\hat{A} = \Sigma$ then $X \overset{d}{=} \mu + \hat{A}'Y$.*

*Proof.* The lemma follows immediately by $\mathbb{E}e^{it'A'Y} = \mathbb{E}e^{i(At)'Y} = \Phi((At)'(At)) = \Phi(t'\Sigma t) = \Phi((\hat{A}t)'(\hat{A}t)) = \mathbb{E}e^{it'\hat{A}'Y}$ and the uniqueness theorem for characteristic functions. □

*Proof (of Theorem 5.2).* Let $X \in E_n(\mu, \Sigma, \Phi)$, $n \geq 2$, with positive definite matrix $\Sigma$ and stochastic representation $X \overset{d}{=} \mu + A'Y \overset{d}{=} \mu + R_n A' U^{(n)}$. According to Lemma 5.3 we may restrict ourself to a 2-dimensional elliptical distribution. Moreover, without loss of generality we set $\mu = 0$, i.e., $X \in E_2(0, \Sigma, \Phi)$. Let $\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$ and $A = \begin{pmatrix} \sqrt{\sigma_{11}} & \sigma_{21}/\sqrt{\sigma_{11}} \\ 0 & \sqrt{\sigma_{22}}\sqrt{1-\rho^2} \end{pmatrix}$ the corresponding Cholesky decomposition, where $\rho := \dfrac{\sigma_{21}}{\sqrt{\sigma_{11}\sigma_{22}}} \in (-1, 1)$, because $\Sigma$ is positive definite. Note that Lemma 5.4 justifies the consideration of this specific type of $A$. Observe that the generalized inverse distribution functions $G_1^{-1}$, $G_2^{-1}$ of $X_1$, $X_2$ are related in the following way: $\sqrt{\sigma_{22}} G_1^{-1}(u) = \sqrt{\sigma_{11}} G_2^{-1}(u)$. The cases where $G_1$ and $G_2$ have a finite right endpoint are ruled out by the regular variation or tail dependence property. Then according to Lemma 5.3

$$\begin{aligned}
&\lim_{u \to 1^-} \mathbb{P}(X_1 > G_1^{-1}(u) \mid X_2 > G_2^{-1}(u)) \\
&\quad = \lim_{x \to \infty} \mathbb{P}(X_1 > \sqrt{\sigma_{11}}x \mid X_2 > \sqrt{\sigma_{22}}x) \\
&\quad = \lim_{x \to \infty} \frac{\mathbb{P}(Y_1 > x, \frac{\sigma_{21}}{\sqrt{\sigma_{11}}} Y_1 + \sqrt{\sigma_{22}}\sqrt{1-\rho^2} Y_2 > \sqrt{\sigma_{22}}x)}{\mathbb{P}(Y_1 > x)} \\
&\quad = \lim_{x \to \infty} \frac{\mathbb{P}(R_2 B_2 > x, R_2 B_2(\rho + \sqrt{1-\rho^2}\sqrt{\frac{1-B_2^2}{B_2^2}} U_2^{(1)}) > x)}{\mathbb{P}(R_2 B_2 > x)},
\end{aligned}$$

where we used the mutual independence of $U_1^{(1)}$, $U_2^{(1)}$, $R_2$, and $B_2$, and applied the fact that $U_1^{(1)}$, and $U_2^{(1)}$ are Bernoulli distributed. Further, straightforward calculations show that

$$\begin{aligned}
&\lim_{u\to 1^-} \mathbb{P}(X_1 > G_1^{-1}(u) \mid X_2 > G_2^{-1}(u)) \\
&= \frac{1}{2}\lim_{x\to\infty}\left(\frac{\mathbb{P}(R_2B_2 > x, R_2B_2(\rho+\sqrt{1-\rho^2}\sqrt{\frac{1-B_2^2}{B_2^2}}) > x)}{\mathbb{P}(R_2B_2 > x)}\right. \\
&\left.+\frac{\mathbb{P}(R_2B_2 > x, R_2B_2(\rho-\sqrt{1-\rho^2}\sqrt{\frac{1-B_2^2}{B_2^2}}) > x)}{\mathbb{P}(R_2B_2 > x)}\right) \\
&= \frac{1}{2}\lim_{x\to\infty}\frac{2\int_0^{h(\rho)}\mathbb{P}(R_2 > \frac{x}{u})\frac{1}{\sqrt{1-u^2}}du}{\int_0^1 \mathbb{P}(R_2 > \frac{x}{u})\frac{1}{\sqrt{1-u^2}}du}
\end{aligned}$$

with $h(\rho) := \left(1+\frac{(1-\rho)^2}{1-\rho^2}\right)^{-1/2}$. The last equality rises from the fact that $B_2^2$ is $Beta(\frac{1}{2},\frac{1}{2})$ distributed, and $B_2$, and $R_2$ are independent random variables. Exploiting exactly the same techniques as in the spherical case for

$$\frac{\int_0^{h(\rho)}\mathbb{P}(R_2 > \frac{x}{u})\frac{1}{\sqrt{1-u^2}}du}{\int_0^1 \mathbb{P}(R_2 > \frac{x}{u})\frac{1}{\sqrt{1-u^2}}du} \qquad \text{instead of} \qquad \frac{\int_0^{\frac{1}{\sqrt{2}}}\mathbb{P}(R_2 > \frac{x}{u})\frac{1}{\sqrt{1-u^2}}du}{\int_0^1 \mathbb{P}(R_2 > \frac{x}{u})\frac{1}{\sqrt{1-u^2}}du}$$

yields the desired result (compare to the proof of Theorem 2.4 and Corollary 4.1). □

**Remark.** Along the lines of the last proof we have not shown that an elliptically contoured random vector possesses the tail dependence property if and only if its corresponding spherical random vector in the sense of (5.1) possesses the tail dependence property. This is still an open question.

**Theorem 5.5.** *Let $X \in E_n(\mu,\Sigma,g)$, $n \geq 2$, with positive definite matrix $\Sigma$ and stochastic representation $X \overset{d}{=} \mu + A'Y$, where $Y \in S_n(g)$ possesses the density generator $g$. Then*
*$\alpha$) all bivariate margins of $X$ possess the tail dependence property if $g$ is regularly varying, i.e., $g \in RV_{-(\alpha+n)/2}$, $\alpha > 0$, and*
*$\beta$) if $X$ possesses a tail dependent bivariate margin and $g$ satisfies Condition 4.3, then $g$ must be O-regularly varying.*

*Proof.* Suppose $X \in E_n(\mu,\Sigma,\Phi)$, $n \geq 2$, with positive definite matrix $\Sigma$ and stochastic representation $X \overset{d}{=} \mu + A'Y \overset{d}{=} \mu + A'R_nU^{(n)}$, where $Y \in S_n(g)$.
$\alpha$) Let $g \in RV_{-(\alpha+n)/2}$, $\alpha > 0$. Then Proposition 3.7 and Karamata's Theorem (see Bingham, Goldie, and Teugels [2], p. 26) imply that $\overline{F}_n \in RV_{-\alpha}$, $\alpha > 0$, where $\overline{F}_n$ denotes the tail function of the generating variate $R_n$ of $Y$. Hence, all bivariate margins of $X$ are tail dependent according to Theorem 5.2.
$\beta$) Assume $X \in E_n(\mu,\Sigma,g)$ possesses a tail dependent bivariate margin. Then, according to Theorem 5.2, the tail function of the univariate margins of $Y$ must be O-regularly varying. Further $g$ satisfies Condition 4.3 and the density function

$g(x^2)$ of $G$ inherits this condition, which yields (4.2) for $G$. Therefore, Lemma 4.2 requires $g$ to be O-regularly varying. □

Finally, we state the closed form expression for the tail dependence coefficient for an elliptically contoured random vector $(X_1, X_2)' \in E_2(\mu, \Sigma, \Phi)$ with positive definite matrix $\Sigma$ having a regular varying generating variate with index $-\alpha < 0$ or having a regular varying density generator with index $-\alpha/2 - 1 < 0$ :

$$\lambda = \frac{\displaystyle\int_0^{h(\rho)} \frac{u^\alpha}{\sqrt{1-u^2}} du}{\displaystyle\int_0^1 \frac{u^\alpha}{\sqrt{1-u^2}} du}, \tag{5.2}$$

with $\rho := \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}$ and $h(\rho) := \left(1 + \frac{(1-\rho)^2}{1-\rho^2}\right)^{-1/2}$ (see also Figure 2). This formula has been developed in the proof of Theorem 5.2. Note, that $\rho$ corresponds to the correlation coefficient in the case of existence (see Fang, Kotz, and Ng [6], p.44, for the covariance formula of elliptically contoured distributions). We remark that the (upper) tail dependence index $\lambda$ coincides with the lower tail dependence index and depends only on the (correlation) coefficient $\rho$ and the regular variation index $\alpha$.

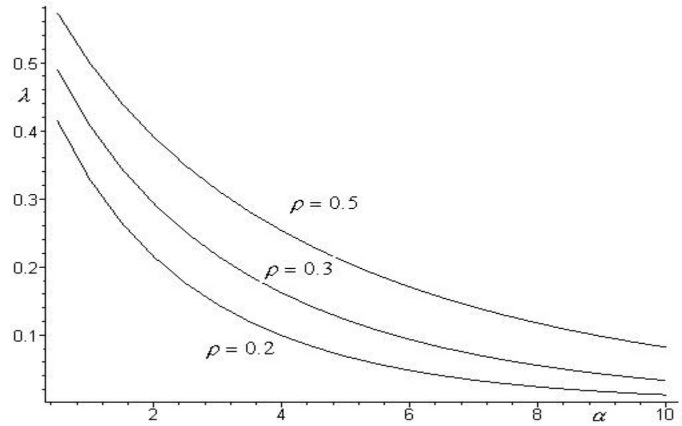


FIGURE 2. Tail dependence index $\lambda$ versus regular variation index $\alpha$ for $\rho = 0.5,\ 0.3,\ 0.1$

## 6. Examples of elliptically contoured distributions

In the following subsections we investigate for several elliptically contoured distributions, whether they possess the tail dependence property or their bivariate marginal distributions are tail independent.

6.1. **Multivariate normal distributions and Kotz type distributions.** Multivariate normal distributions are included in the class of symmetric Kotz type distributions. Therefore, we restrict our attention to the latter family of distributions.

**Definition 6.1.** Let $X \in E_n(\mu, \Sigma, g)$. Then we call $X$ symmetric Kotz type distributed if the density generator $g$ is of the form

$$g(u) = C_n u^{N-1} \exp(-ru^s), \quad r, s > 0,\ 2N + n > 2, \tag{6.1}$$

where $C_n$ is a normalizing constant.

**Theorem 6.2.** *Let $X \in E_n(\mu, \Sigma, g)$, $n \geq 2$, be a symmetric Kotz type distributed random vector. Then all bivariate margins of $X$ are tail independent.*

*Proof.* Observe that the density generator (6.1) does not belong to the class of $O$-regularly varying functions, because

$$\lim_{u\to\infty} \frac{g(tu)}{g(u)} = \lim_{u\to\infty} t^{N-1} \exp(-ru^s(t^s - 1)) = 0$$

for all $t > 1$, $r, s > 0$, and $2N+n > 2$. Therefore Theorem 5.5 yields the conclusion, since the prerequisites are fulfilled. □

**Remark.** The density generator (6.1) belongs to the multivariate normal distribution if $N = s = 1$ and $r = 1/2$.

6.2. **Multivariate t-distributions and symmetric Pearson type VII distributions.** Multivariate t-distributions are included in the class of symmetric Pearson type VII distributions. Thus, we investigate again the larger class for the tail dependence property.

**Definition 6.3.** Let $X \in E_n(\mu, \Sigma, g)$. Then we call $X$ symmetric Pearson type VII distributed if its density generator is of the form

$$g(u) = C_n (1 + \frac{u}{m})^{-N}, \quad N > n/2,\ m > 0, \tag{6.2}$$

where $C_n$ denotes a normalizing constant.

**Theorem 6.4.** *Let $X \in E_n(\mu, \Sigma, g)$, $n \geq 2$, be a symmetric Pearson type VII distributed random vector. Then all bivariate margins of $X$ possess the tail dependence property.*

*Proof.* Obviously the density generator (6.2) is regularly varying with index $-N$, and the conclusion follows with Theorem 5.5. □

**Remark.**
Setting $N = (n+m)/2$ and $m \in \mathbb{N}$ in (6.2) yields the density generator for the well-known class of multivariate t-distributions, which includes the multivariate Cauchy distribution for $m = 1$.

6.3. **Multivariate logistic distribution.**

**Definition 6.5.** We call $X \in E_n(\mu, \Sigma, g)$ a multivariate logistically distributed random vector if its density generator is given by

$$g(u) = C_n \exp(-u)/(1 + \exp(-u))^2,$$

where $C_n$ is a normalizing constant.

**Theorem 6.6.** *Suppose $X \in E_n(\mu, \Sigma, g)$, $n \geq 2$, is a logistically distributed random vector then all bivariate margins of $X$ are tail independent.*

*Proof.* First, observe that $g'(u) = C_n \exp(-u)/(1+\exp(-u))^2(\frac{2\exp(-u)}{1-\exp(-u)} - 1) < 0$ for all $u > \ln 2$. Further

$$\lim_{u\to\infty} \frac{g(tu)}{g(u)} = \lim_{u\to\infty} \exp(-u(t-1))\Big(\frac{1+\exp(-u)}{1+\exp(-tu)}\Big)^2 = 0$$

for all $t > 1$ and and therefore Theorem 5.5 yields the result. □

### 6.4. **Multivariate symmetric generalized hyperbolic distributions.**

**Definition 6.7.** We call $X \in E_n(\mu, \Sigma, g)$ a multivariate symmetric generalized hyperbolic distributed random vector if its density generator is given by

$$g(u) = C_n \frac{K_{\lambda-\frac{n}{2}}(\sqrt{\psi(\chi+u)})}{(\sqrt{\chi+u})^{\frac{n}{2}-\lambda}},\ u > 0, \tag{6.3}$$

where $\psi, \chi > 0,\ \lambda \in I\!R,\ C_n$ is a normalizing constant, and $K_\nu$ denotes the modified Bessel function of the third kind (or MacDonald function).

**Theorem 6.8.** *Let $X \in E_n(\mu, \Sigma, g)$, $n \ge 2$, be a multivariate symmetric generalized hyperbolic distribution. Then all bivariate margins of $X$ are tail independent.*

*Proof.* We show that the density generator (6.3) is monotone decreasing. Applying the following relationships for modified Bessel functions of the third kind

$$\frac{d}{dx}K_\nu(x) = -K_{\nu-1}(x) - \frac{\nu}{x}K_\nu(x), \quad K_\nu = K_{-\nu},$$

and $K_\nu(x) > 0$ for all $x \ge 0$ we obtain

$$\frac{d}{dx}K_{-\nu}(x) = \frac{d}{dx}K_\nu(x) = -K_{\nu-1}(x) - \frac{\nu}{x}K_\nu(x) < 0$$

for all $\nu \ge 0$ and $x \ge 0$. Hence, $g$ is monotone decreasing. Further,

$$\begin{aligned}\lim_{u\to\infty}\frac{g(tu)}{g(u)} &= \lim_{u\to\infty}\frac{K_{\lambda-\frac{n}{2}}(\sqrt{\psi(\chi+tu)})}{K_{\lambda-\frac{n}{2}}(\sqrt{\psi(\chi+u)})}\left(\sqrt{\frac{\lambda+u}{\lambda+tu}}\right)^{\frac{n}{2}-\lambda}\\ &= (\sqrt{t})^{\lambda-\frac{n}{2}}\lim_{u\to\infty}\frac{K_{\lambda-\frac{n}{2}}(\sqrt{\psi(\chi+tu)})}{K_{\lambda-\frac{n}{2}}(\sqrt{\psi(\chi+u)})} = 0\end{aligned}$$

for all $t > 1$, according to the asymptotic behavior $K_\nu \sim \sqrt{\frac{\pi}{2x}}e^{-x}(1+o(1))$ (see Abramowitz and Stegun [1], p. 378, Formula 9.7.2), and thus

$$\lim_{x\to\infty}\frac{K_\nu(sx)}{K_\nu(x)} = \lim_{x\to\infty}\frac{1}{\sqrt{s}}e^{-x(s-1)} = 0 \qquad \forall s > 1.$$

□

## 7. Other dependence measures for elliptically contoured distributions

In this section we turn to other dependence measures for elliptically contoured distributions which might be useful in the study of credit risk models. For instance we might be interested in a dependence measure which describes the dependence structure of absolute asset returns. Then we could consider the following dependence measure for elliptically contoured random vectors $(X_1, X_2)' \in E_2(\mu, \Sigma, \Phi)$

$$\hat{\lambda} := \lim_{x\to G_2^{-1}(1)^-} I\!P(X_1 > x \mid X_2 > x),$$

if the limit exists, where $G_2^{-1}$ denotes the generalized inverse distribution function of $X_2$. We call $\hat{\lambda}$ the extreme dependence index for absolute values and say $(X_1, X_2)'$ is extreme absolute value dependent if $\hat{\lambda} > 0$ and independent if $\hat{\lambda} = 0$.
Analogous to Theorem 5.2 and Theorem 5.5, related to the tail dependence, we can establish similar results for extreme absolute value dependence. In particular, in the case of a bivariate elliptically contoured random vector $X \in E_2(\mu, \Sigma, g),\ n \geq 2$, $\Sigma$ positive definite, with regularly varying density generator with index $-\alpha-1,\ \alpha > 0$, we derive

$$\hat{\lambda} = \frac{\frac{1}{2}\int_{h_1(\Sigma)}^{\infty} \frac{u^{-\alpha-1}}{\sqrt{u^2-1}} du + \frac{1}{2}\int_{h_2(\Sigma)}^{\infty} \frac{u^{-\alpha-1}}{\sqrt{u^2-1}} du}{\int_1^{\infty} \frac{u^{-\alpha-1}}{\sqrt{u^2-1}} du}, \tag{7.1}$$

where $\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \in I\!R^{2\times 2}$,

$$g_1(\Sigma) = \sqrt{1 + \frac{(\sigma_{22} - \sigma_{12})^2}{\sigma_{11}\sigma_{22} - \sigma_{12}^2}}, \quad \text{and} \quad g_2(\Sigma) = \sqrt{1 + \frac{(\sigma_{11} - \sigma_{12})^2}{\sigma_{11}\sigma_{22} - \sigma_{12}^2}}.$$

In the following figure we illustrate that this dependence measure depends on the individual volatilities of $X_1$ and $X_2$.

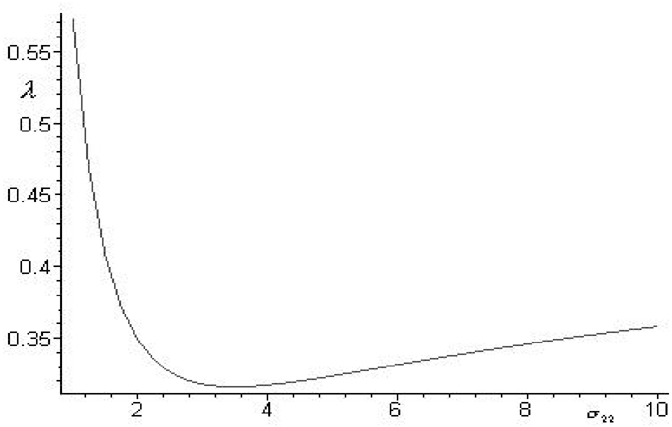


FIGURE 3. Extreme dependence index $\hat{\lambda}$ versus volatility $\sigma_{22}$ for regular variation index $\alpha = 2.5$ and $\sigma_{11} = 2$

Further, one might be interested in a multidimensional extension of the bivariate tail dependence concept defined in this work. The following definition gives one possible approach.

**Definition 7.1.** Let $X$ be a $n$-dimensional random vector. We say that $X$ is *multivariate tail dependent* if for some sets $I \cup J = \{1, \ldots, n\}$, and $I \cap J = \emptyset$ the following limit exists:

$$\lambda := \lim_{u \to 1^-} I\!P(X_i > G_i^{-1}(u),\ \forall i \in I \mid X_j > G_j^{-1}(u),\ \forall j \in J) > 0; \tag{7.2}$$

where $G_i^{-1},\ G_j^{-1}$, denote the generalized inverse distribution functions of $X_i,\ X_j$. Consequently, we say $X$ is *multivariate tail independent* if $\lambda$ equals 0. Further, we call $\lambda$ the (upper) multivariate tail dependence coefficient.

Results related to the above multivariate extension of tail dependence will be presented in a forthcoming work.

## Conclusions

Summarizing the results we have found an appealing way of characterizing tail dependent elliptically contoured distributions by regular and O-regular tail properties of the corresponding one-dimensional distribution functions or generating distributions. We applied the above results to a number of well-known elliptically contoured distributions in order to find out whether they are tail dependent or not. In this framework we showed that the symmetric Pearson type VII distributions (including the multidimensional t-distributions) have the tail dependence property. Therefore, there are a number of elliptically contoured distributions which inherit many useful properties of the multidimensional normal distribution and moreover have additional necessary properties to model credit risk in a more realistic way. Due to the existence of good estimation and simulation techniques for both distributions mentioned above their usage is favorable for dependence modelling within credit risk models.

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Universität Ulm
Abteilung Zahlentheorie und Wahrscheinlichkeitstheorie
Helmholtzstr. 18
89069 Ulm, Germany

*E-mail address*: Rafael.Schmidt@mathematik.uni-ulm.de